# Math 239 — Sample Questions for Hour Exam 2

Spring, 2018

This document is a collection of questions relevant to our upcoming exam that I have used on past Proofs exams. I’ve included the original point value of each question, as an indication of how “big” I think each is (our exam will have a total of 50 points). All of the questions address material that might appear on our exam, but there are more questions here than will appear on it. I’ve included my solutions to each question, but I strongly recommend that you try to answer each question for yourself before looking at the solutions. My proofs italicize variable names, since I am typing my solutions, whereas you can write yours by hand and will not need to distinguish italic from regular characters.

**Question 1** (15 Points). Prove that the number 5 - √2 is irrational. Your proof should follow our conventions for formal proof-writing, except for conventions that require using italics or other typefaces that would be hard to achieve with handwriting.

**Solution:** We prove by contradiction that 5 - √2 is irrational. Assume for the sake of contradiction that 5 - √2 is rational, i.e., there exist integers *a* and *b*, with *b* ≠ 0, such that 5 - √2 = *a*/*b*. Algebraically rearranging this equation yields √2 = 5 – *a*/*b*, which is rational since the rationals are closed under subtraction. But this is a contradiction, because we have previously proven that √2 is irrational. We have thus proven by contradiction that 5 - √2 must be irrational. QED.

**Question 2** (15 Points). Let the function *B*(*x*) be defined as

$$B\left(x\right)= \left\{\begin{matrix}x&If 0\leq x<1\\2-x&If 1\leq x\leq 2\\0&Otherwise\end{matrix}\right.$$

(*B*(*x*) is a “first-degree basis spline.” It’s not very useful in its own right, but is a simple member of a family of functions that are widely used for approximating complicated curves with low-degree polynomials, among other things.)

Prove that for all real numbers *x*, 0 ≤ *B*(*x*) ≤ 1. Your proof should follow our conventions for formal proof-writing, except for conventions that require using italics or other typefaces that would be hard to achieve with handwriting.

**Solution:** We use cases to prove that for all real numbers *x*, 0 ≤ *B*(*x*) ≤ 1. The cases correspond to the cases in the definition of *B*(*x*), namely…

Case 1: 0 ≤ *x* < 1. In this case *B*(*x*) = *x*, and so 0 ≤ *B*(*x*) < 1 which in turn means
0 ≤ *B*(*x*) ≤ 1 .

Case 2: 1 ≤ *x* ≤ 2. In this case *B*(*x*) = 2 – *x*. Since 1 ≤ *x* ≤ 2, we have 2 – 1 ≥ 2 – *x* ≥ 2 – 2, or 0 ≤ 2 – *x* ≤ 1. Since *B*(*x*) = 2 – *x*, these last inequalities show that 0 ≤ *B*(*x*) ≤ 1.

Case 3: *x* < 0 or *x* > 2. In this case *B*(*x*) is defined by be 0, so 0 ≤ *B*(*x*) ≤ 1.

These three cases cover all possible real values for *x,* and so together prove that for all real numbers *x*, 0 ≤ *B*(*x*) ≤ 1. QED.

**Question 3** (15 Points). Prove that if *a* and *b* are integers and the product *ab* is not divisible by 9, then at least one of *a* and *b* is not divisible by 3.

**Solution:** We prove that if *a* and *b* are integers and the product *ab* is not divisible by 9, then at least one of *a* and *b* is not divisible by 3 by proving the contrapositive. In other words, we prove that if integers *a* and *b* are both divisible by 3, then *ab* is divisible by 9. Assume that *a* and *b* are integers divisible by 3, i.e., that there exist integers *x* and *y* such that *a* = 3*x* and *b* = 3*y*. We can then write the product *ab* as

$$ab=(3x)(3y)$$

$$=9xy$$

Since the integers are closed under multiplication, *xy* is an integer and so *ab* is a multiple of 9. We have thus proven the contrapositive, and so have also proven that if *a* and *b* are integers and the product *ab* is not divisible by 9, then at least one of *a* and *b* is not divisible by 3. QED.

**Question 4** (20 Points). Prove that for all natural numbers *n*, $11^{n} $is odd.

**Solution:** We use induction on *n* to prove that for all natural numbers *n*, $11^{n} $is odd.

For the base case, consider *n* = 1. 111 = 11, which is odd.

For the induction step, we show that if *k* is a natural number such that 11*k* is odd, then 11*k*+1 is also odd. Assume that that 11*k* is odd, and note that 11*k*+1 = 11 × 11*k*. We know that both 11 and 11*k* are odd, and that the product of two odd numbers is odd. Thus we conclude that 11*k*+1 is odd.

We have now established both the base case and the induction step, and so have shown by mathematical induction that for all natural numbers *n*, $11^{n} $is odd. QED.

**Question 5** (15 Points). Prove that no integer *n* has the property that *n* ≡ 1 (mod 2) and
*n* ≡ 4 (mod 6).

**Solution:** We prove by contradiction that no integer *n* has the property that
*n* ≡ 1 (mod 2) and *n* ≡ 4 (mod 6). Assume for the sake of contradiction that *n* is an integer and that *n* ≡ 1 (mod 2) and *n* ≡ 4 (mod 6). Since *n* ≡ 1 (mod 2), we know that there is some integer *x* such that *n* = 2*x* + 1. Similarly, there is some integer *y* such that
*n* = 6*y* + 4. Since these expressions both equal *n*, we have an equality that we can rearrange as follows:

$$2x+1=6y+4$$

$$2x-6y=3$$

$$2\left(x-3y\right)=3$$

Since the integers are closed under multiplication and subtraction, *x* – 3*y* is an integer, and so we have shown that 3 is even. But this is a contradiction, since 3 is clearly odd. We thus conclude that no integer *n* has the property that *n* ≡ 1 (mod 2) and
*n* ≡ 4 (mod 6). QED.