Math 239 — Final Exam Part 1

May 10, 2017

Directions. During this part of the exam you may not use any references or communicate with any person except me. When you finish this part, turn in this page and continue with Part 2 under its rules. Put your answer to the question in the space provided (use the back of the page if you need more space). If you can't answer completely, say as much as you can — I give partial credit for incorrect answers if you show correct steps leading up to them. Good luck.

This part contains 1 question on 1 page.

Question 1 (20 Points). Let A, B, and C be nonempty sets. Furthermore, let $f : A \to B$ and $g : B \to C$ be functions. Write a formal proof that if f and g are both injections, then $g \circ f$ is an injection.

> **Proof:** We prove that if A, B, and C are nonempty sets, and $f : A \to B$ and $g : B \to C$ are injections, then $g \circ f$ is an injection by showing that for all elements x and y of A, if $(g \circ f)(x) = (g \circ f)(y)$, then $x = y$. So assume that $(g \circ f)(x) = (g \circ f)(y)$, i.e., that $g(f(x)) = g(f(y))$. Since g is an injection, we must have $f(x) = f(y)$. And now, since f is also an injection, x must equal y. We have thus proved that if A, B, and C are nonempty sets, and $f : A \to B$ and $g : B \to C$ are injections, then $g \circ f$ is an injection. QED.

Math 239 — Final Exam Part 2

May 10, 2017

General Directions. This part of the exam is open book, open notes, and open computer. However, you may not communicate with any person, except me, during the test. You have a total of 2 ½ hours in which to do this part and Part 1. Put your answer to each question in the space provided (use the backs of pages if you need more space). Be sure to **show your work**! I give partial credit for incorrect answers if you show correct steps leading up to them; conversely, I do not give full credit even for correct answers if it is not clear that you understand where those answers come from. Good luck.

This part contains 5 questions, numbered questions 2 through 6, on 5 pages.

Question 2 (20 Points). Define the Math 239 Numbers to be the members of the set

$$\left\{\frac{n-239}{n} \middle| n \in \mathbb{N}\right\}$$

Determine whether the set of all Math 239 Numbers is finite, countably infinite, or uncountably infinite, and support your answer with a formal proof.

Proposition. The set of Math 239 numbers is countably infinite.

Proof. We show that the set of Math 239 numbers is countably infinite by showing a bijection between it and the natural numbers. In particular, the function

$$f(n) = \frac{n-239}{n}$$

is a bijection from the natural numbers to the Math 239 numbers, and so f^{-1} is a bijection from the Math 239 numbers to the naturals. To see that f is an injection, suppose that $f(n) = f(m)$ for natural numbers n and m. Then

$$\frac{n-239}{n} = \frac{m-239}{m}$$

and so

$$mn - 239m = mn - 239n$$

or

$$239m = 239n$$

and so $m = n$. To see that f is a surjection, note that f implements the definition of the Math 239 numbers, and so every Math 239 number is necessarily the image of some natural under f. We have thus shown that there is a bijection between the set of Math 239 numbers and the natural numbers, and so the set of Math 239 numbers is countably infinite. QED.

Question 3 (15 Points). Define two points in the plane to be "about the same" if the distance between them is small. More precisely, define "about the same" to be a relation on ordered pairs of reals such that (x_1,y_1) is about the same as (x_2,y_2) if and only if

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ \leq 1$$

Is "about the same" an equivalence relation? If so, give a formal proof; if not identify which properties of an equivalence relation it has and which it does not have, explaining but not formally proving each claim in a sentence or two.

"About the same" is reflexive and symmetric, but not transitive, and so it is not an equivalence relation.

"About the same" is reflexive because for every point (x,y),

$$\sqrt{(x - x)^2 + (y - y)^2} = 0 \leq 1$$

and so (x,y) is about the same as itself.

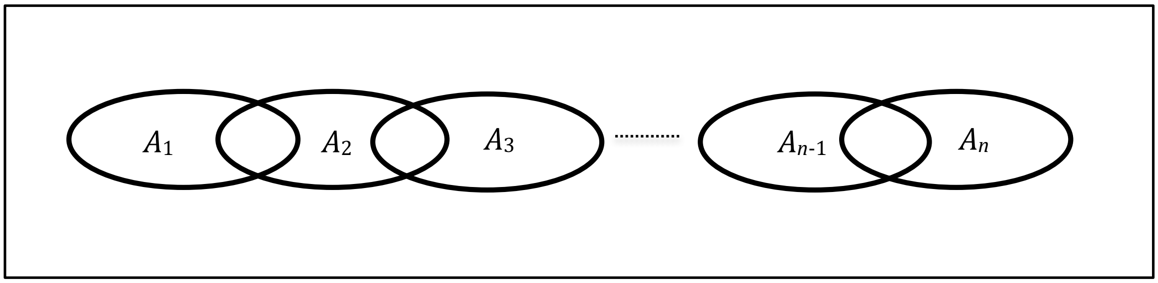
"About the same" is also symmetric, because for every pair of points $(x_1,y_1,)$ and (x_2,y_2),

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

and so $(x_1,y_1,)$ is about the same as (x_2,y_2) if and only if $(x_2,y_2,)$ is about the same as (x_1,y_1).

However, "about the same" is not transitive because, for example (0,0) is about the same as (0,1), and (0,1) is about the same as (0,2), but (0,0) is not about the same as (0,2).

Question 4 (15 Points). Let $n \geq 3$ be a natural number and let $\mathcal{A} = \{ A_i \mid 1 \leq i \leq n \}$ be a family of sets. Informally, let each set in $\mathcal{A}$ have a non-empty intersection with the next and previous members of $\mathcal{A}$, but no other intersections, as in this Venn diagram:



Formally, for all natural numbers i such that $1 \leq i < n$, $A_i \cap A_{i+1} \neq \emptyset$, but for all natural numbers j and k such that $1 \leq j < k\text{-}1 < n$, $A_j \cap A_k = \emptyset$. Give a formal proof that under these conditions,

$$\bigcap_{1 \leq i \leq n} A_i = \emptyset$$

Proof. We prove by contradiction that under the conditions given in the question, $\bigcap_{1 \leq i \leq n} A_i = \emptyset$. Assume for the sake of contradiction that the intersection is not empty. Then there is some element, call it x, that is a member of the intersection, and hence of every member of $\mathcal{A}$. So in particular, $x \in A_1$ and $x \in A_3$, contradicting the requirement that for all natural numbers j and k such that $1 \leq j < k\text{-}1 < n$, $A_j \cap A_k = \emptyset$ (specifically, the contradiction happens when $j = 1$ and $k = 3$, which is always possible since $n \geq 3$). We have thus shown that it is impossible for $\bigcap_{1 \leq i \leq n} A_i$ to not be empty, and so that $\bigcap_{1 \leq i \leq n} A_i = \emptyset$. QED.

Question 5 (15 Points). Let $f : \mathbb{Z}^+ \to \mathbb{N}$ (where $\mathbb{Z}^+$ denotes the non-negative integers) be defined piecewise as follows:

$$f(n) = \begin{cases} 1 \text{ if } n = 0 \\ (f(n-1))^5 \text{ if } n > 0 \end{cases}$$

Write a formal proof that for all non-negative integers n, $f(n) = 1$.

Proof. We prove by induction on n that $f(n) = 1$ for all non-negative integers n.

For the basis step, let $n = 0$. Then from the definition of f, $f(n) = f(0) = 1$.

For the induction step, we show that if $f(k) = 1$ for some non-negative integer k, then $f(k+1) = 1$. So assume that k is a non-negative integer such that $f(k) = 1$, and consider $f(k+1)$. Since k is a non-negative integer, $k + 1 \geq 1$, and so $f(k+1)$ is defined as

$$\begin{aligned} f(k+1) &= f\big((k+1)-1\big)^5 \\ &= f(k)^5 \\ &= 1^5 \\ &= 1 \end{aligned}$$

Since we have shown that $f(0) = 1$ and that whenever $f(k) = 1$, $f(k+1)$ also equals 1, it follows by the principle of mathematical induction that $f(n) = 1$ for all non-negative integers n. QED.

Question 6 (15 Points). Consider the statement "every city has some neighborhood where everyone is rich, and some neighborhood where everyone is poor." This statement can be written in symbolic logical form as

$$(\forall c \in C)\Big((\exists n \in N)\big((\forall x \in P)(L(x,n) \rightarrow R(x))\big) \wedge (\exists m \in N)\big((\forall y \in P)(L(y,m) \rightarrow P(y))\big)\Big)$$

Where

- C is a universal set of cities
- N is a universal set of neighborhoods
- P is a universal set of people
- $L(x,y)$ is a predicate meaning that person x lives in neighborhood y
- $R(x)$ is a predicate meaning that person x is rich
- $P(x)$ is a predicate meaning that person x is poor.

Derive the symbolic negation of the statement and paraphrase that negation in English. If you need the negation of some predicate, e.g., $Q(x)$, in your answer, just write "$\neg Q(x)$," don't invent a new predicate that means the opposite of Q.

Solution:

$$\neg(\forall c \in C)\Big((\exists n \in N)\big((\forall x \in P)(L(x,n) \rightarrow R(x))\big) \wedge (\exists m \in N)\big((\forall y \in P)(L(y,m) \rightarrow P(y))\big)\Big)$$

$$\equiv (\exists c \in C)\Big(\neg\Big((\exists n \in N)\big((\forall x \in P)(L(x,n) \rightarrow R(x))\big) \wedge (\exists m \in N)\big((\forall y \in P)(L(y,m) \rightarrow P(y))\big)\Big)\Big)$$

$$\equiv (\exists c \in C)\Big(\neg(\exists n \in N)\big((\forall x \in P)(L(x,n) \rightarrow R(x))\big) \vee \neg(\exists m \in N)\big((\forall y \in P)(L(y,m) \rightarrow P(y))\big)\Big)$$

$$\equiv (\exists c \in C)\Big((\forall n \in N)\big(\neg(\forall x \in P)(L(x,n) \rightarrow R(x))\big) \vee (\forall m \in M)\big(\neg(\forall y \in P)(L(y,m) \rightarrow P(y))\big)\Big)$$

$$\equiv (\exists c \in C)\Big((\forall n \in N)\Big((\exists x \in P)\big(\neg(L(x,n) \rightarrow R(x))\big)\Big) \vee (\forall m \in N)\Big((\exists y \in P)\big(\neg(L(y,m) \rightarrow P(y))\big)\Big)\Big)$$

$$\equiv (\exists c \in C)\Big((\forall n \in N)\big((\exists x \in P)(L(x,n) \wedge \neg R(x))\big) \vee (\forall m \in N)\big((\exists y \in P)(L(y,m) \wedge \neg P(y))\big)\Big)$$

In English, this would be "in some city, either every neighborhood has a resident who is not rich, or every neighborhood has a resident who is not poor."