Math 239 Problem Set 9 Solution

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Here are my solutions to the questions in Problem Set 9.

Problem 1. The proposition arising from problems 4a, b, and c in section 4.2 of Sundstrom's text is

Proposition 1. For all natural numbers $n \geq 2$,

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n} \tag{1}$$

Proof. We prove Equation 1 by induction on n.

For the basis step, assume $n = 2$. We then have

$$\begin{aligned}\prod_{i=2}^{2}\left(1-\frac{1}{i^2}\right) &= 1-\frac{1}{4} \\ &= \frac{3}{4} \\ &= \frac{2+1}{2\times 2}\end{aligned}$$

establishing that Equation 1 holds for $n = 2$.

For the induction step, we let $k \geq 2$ be a natural number such that Equation 1 holds when $n = k$, and we show that Equation 1 also holds when $n = k + 1$. We do this by breaking the product on the left side of Equation 1 into two parts and applying the induction hypothesis:

$$\begin{aligned}
\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right) &= \prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)\left(1-\frac{1}{(k+1)^2}\right) \\
&= \frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right) \\
&= \frac{k+1}{2k}\left(\frac{(k+1)^2-1}{(k+1)^2}\right) \\
&= \frac{k+1}{2k}\left(\frac{k^2+2k}{(k+1)^2}\right) \\
&= \frac{1}{2}\left(\frac{k+2}{k+1}\right) \\
&= \frac{(k+1)+1}{2(k+1)}
\end{aligned}$$

We have thus shown that for all natural numbers $k \geq 2$, if Equation 1 holds when $n = k$ then Equation 1 holds when $n = k + 1$.

Having shown both a basis step of $n = 2$ and the induction step, we conclude by the extended principle of mathematical induction that Equation 1 holds for all natural numbers greater than or equal to 2. □

Problem 2. By trying various small natural numbers, it appears that all natural numbers greater than or equal to 12 can be written as a sum of the form $4x + 5y$ for non-negative integers x and y. I thus conjecture the following:

Proposition 2. For all natural number $n \geq 12$, there exist non-negative integers x and y such that $n = 4x + 5y$.

Proof. We let $P(n)$ be the predicate "$n = 4x+5y$ for some non-negative integers x and y," and prove by strong induction on n that P holds for all natural numbers greater than or equal to 12.

We use four basis steps, namely $n = 12$, $n = 13$, $n = 14$, and $n = 15$. For each of these values, we observe

$$\begin{aligned}
12 &= 4 \times 3 + 5 \times 0 \\
13 &= 4 \times 2 + 5 \times 1 \\
14 &= 4 \times 1 + 5 \times 2 \\
15 &= 4 \times 0 + 5 \times 3
\end{aligned}$$

This establishes that P holds for 12, 13, 14, and 15.

For the inductive step, we show that if $k \geq 15$ is a natural number and $P(m)$ is true for all natural numbers m satisfying $12 \leq m \leq k$, then $P(k+1)$ is also true. To show this, notice that because $k \geq 15$, $k-3 \geq 12$; $k-3$ is also less than or equal to k. Thus $P(k-3)$ holds, i.e., $k-3=4x+5y$ for some non-negative integers x and y. But now we see that

$$\begin{aligned} k+1 &= (k-3)+4 \\ &= 4x+5y+4 \\ &= 4(x+1)+5y \end{aligned}$$

Since the non-negative integers are closed under addition, $x+1$ and y are both non-negative integers. Thus $P(k+1)$ is true.

We have thus established, by the second principle of mathematical induction, that for all natural number $n \geq 12$, there exist non-negative integers x and y such that $n = 4x+5y$. □

Problem 3. (Find $A \cap B$, $A \cup B$, and $(A \cup B)^C$ if A is the set of natural numbers greater than or equal to 7, B is the set of odd natural numbers, and the universal set is $\mathbb{N}$.)

$A \cap B$ is the set of natural numbers that are greater than or equal to 7 and odd, i.e., $\{7, 9, 11, \ldots\}$.

$A \cup B$ is the set of natural numbers that are greater than or equal to 7 or odd, i.e., $\{1, 3, 5, 7, 8, 9, 10, \ldots\}$.

$(A \cup B)^C$ is the set of natural numbers that are neither greater than or equal to 7 nor odd. By de Morgan's Law, this is the set of natural numbers less than 7 and even, or $\{2, 4, 6\}$.

Problem 4. Using results stated or proven in Problem Set 5, prove the following:

Proposition 3. For any real numbers x and y such that $x < y$, there is a natural number n such that $ny - nx > 1$.

Proof. We prove the proposition using a direct proof from Problem Set 5's proposition 1 (which said that for all real numbers x, if $x > 0$ then there is a natural number n such that $0 < \frac{1}{n} < x$). Since $x < y$, $y - x > 0$ and so by Problem Set 5's Proposition 1, there exists a natural number n such that $0 < \frac{1}{n} < y - x$. Multiplying both sides of the second inequality by n yields $1 < ny - nx$, or $ny - nx > 1$. We have thus proven that for any real numbers x and y such that $x < y$, there is a natural number n such that $ny - nx > 1$. □