Math 239 Problem Set 4 Solution

February 9, 2017

Here are my solutions to the questions in Problem Set 4.

Problem 1. (Use the roster method to specify the set of integers n such that $\sqrt{n}$ is a natural number and $n < 50$.)

Since the square roots are natural numbers, they must be 1, 2, 3, etc. Squaring successive naturals until the square equals or exceeds 50 gives the required set, namely $\{1, 4, 9, 16, 25, 36, 49\}$.

Problem 2. (Use set builder notation to specify the set of all integers that are multiples of 5.)

All multiples of 5 are of the form $5k$ for some integer k, and every number of that form is a multiple of 5. Putting these ideas in set builder notation yields $\{5k | k \in \mathbb{Z}\}$.

Problem 3. (Describe and give examples of the truth set for the predicate "$|x| < \pi$")

Note that the truth set is a set of values that make the predicate true. One way to write it with set builder notation is $\{x \in \mathbb{R} | -\pi < x < \pi\}$. Two members of the truth set are 0 and 1; two real numbers not in the truth set are -10 and 10.

Problem 4. (Describe sets in English and roster notation.)

The set $\{x \in \mathbb{R} | x^2 = 16\}$ is the set of (real) square roots of 16, i.e., the set $\{-4, 4\}$.

The set $\{x \in \mathbb{R} | x^2 + 16 = 0\}$ is the set of real square roots of -16. Since there are no such real numbers, this is the empty set, $\emptyset$ or $\{\}$.

Problem 5. (Determine the equality or subset relationships in the following pairs of sets.)

$\{4n + 1 | n \in \mathbb{Z}\}$ and the odd integers. Every member of $\{4n + 1 | n \in \mathbb{Z}\}$ is odd, so $\{4n + 1 | n \in \mathbb{Z}\}$ is a subset of the odd integers. But there are odd integers that are not members of $\{4n + 1 | n \in \mathbb{Z}\}$, for example 3 (which would require $n = \frac{1}{2}$, which is not an integer, to be written in the form $4n + 1$), so $\{4n + 1 | n \in \mathbb{Z}\}$ is not equal to the odd integers.

$\mathbb{R}$ and $\{x \in \mathbb{R} | \sqrt{x} \in \mathbb{Z}\}$. By construction, $\{x \in \mathbb{R} | \sqrt{x} \in \mathbb{Z}\}$ is a subset of $\mathbb{R}$, so the only question is whether the two sets are equal. Since there are real numbers whose square roots are not integers (for example 2), the sets are not equal.

The set of prime numbers and $\{7n | n \in \mathbb{N} \wedge n \geq 2\}$. Every number of the form $7n$ for some natural number n greater than or equal to 2 has at least 7 and n as factors, and so cannot be prime. Conversely, no prime number can have 7 and a natural number greater than or equal to 2 as factors, so these two sets are completely disjoint (have no members in common).

Problem 6. (Determine whether the set $A = \{1, 4, 7, 10, 13, \ldots\}$ is closed under addition and multiplication, giving formal proofs for affirmative answers and counterexamples for negative ones.)

Set A is not closed under addition, since 1 and 4 are both members of A but $1 + 4 = 5$ is not.

Set A is closed under multiplication, as shown by the following proposition:

Proposition 1. If x and y are elements of set A, then xy is also an element of A.

Proof. We let x and y be elements of A, and will show by a direct proof that xy is also an element of A. Note that the pattern that defines the members of A is that every member is of the form $3k + 1$ for some integer $k \geq 0$. Thus there must be non-negative integers m and n such that $x = 3m + 1$ and $y = 3n + 1$. We then see that

$$\begin{aligned} xy &= (3m+1)(3n+1) \\ &= 9mn + 3m + 3n + 1 \\ &= 3(3mn + m + n) + 1 \end{aligned}$$

The non-negative integers are closed under addition and multiplication, so $3mn + m + n$ is a non-negative integer, which in turn means that xy is in A. □