Math 239 Problem Set 11 Solution

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Here are my solutions to the questions in Problem Set 11.

Problem 1. (Find the Cartesian product $\{-1,0,1\} \times \{a,b\}$.)

The Cartesian product is the set of all ordered pairs that can be made from the two sets, so

$$\{-1,0,1\} \times \{a,b\} = \{(-1,a),(-1,b),(0,a),(0,b),(1,a),(1,b)\}$$

Problem 2.

Proposition 1. If A and B are nonempty sets, then $A \times B = B \times A$ if and only if $A = B$.

Proof. We prove that if A and B are nonempty sets, then $A \times B = B \times A$ if and only if $A = B$ by proving each direction separately.

First, we show that if $A \times B = B \times A$ then $A = B$. From the definition of Cartesian product, we know that for every x that is in A and every y in B, (x,y) is in $A \times B$. Since $A \times B = B \times A$, (x,y) must also be a member of $B \times A$, which means that $x \in B$ and $y \in A$. Thus we see that every member x of A is in B, and every member y of B is in A, and so $A = B$.

For the second direction, we show that if $A = B$, then $A \times B = B \times A$. So assume $A = B$, and notice that substituting equals for equals then yields

$$\begin{aligned} A \times B &= A \times A \\ &= B \times A. \end{aligned}$$

We have thus shown that if $A = B$, then $A \times B = B \times A$.

Since we have now established implications in both directions, we see that if A and B are nonempty sets, then $A \times B = B \times A$ if and only if $A = B$. □

Problem 3.

Proposition 2. If Λ is a nonempty indexing set and $\mathcal{A} = \{A_\alpha | \alpha \in \Lambda\}$ is an indexed family of sets, then

$$\left(\bigcup_{\alpha \in \Lambda} A_\alpha \right)^C = \bigcap_{\alpha \in \Lambda} A_\alpha^C$$

Proof. We prove that if Λ is a nonempty indexing set and $\mathcal{A} = \{A_\alpha | \alpha \in \Lambda\}$ is an indexed family of sets, then $(\bigcup_{\alpha\in\Lambda} A_\alpha)^C = \bigcap_{\alpha\in\Lambda} A_\alpha^C$ by proving

$$\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^C \subseteq \bigcap_{\alpha\in\Lambda} A_\alpha^C \tag{1}$$

and

$$\bigcap_{\alpha\in\Lambda} A_\alpha^C \subseteq \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^C \tag{2}$$

To prove relationship 1, let x be any member of $(\bigcup_{\alpha\in\Lambda} A_\alpha)^C$. Thus x is not in $\bigcup_{\alpha\in\Lambda} A_\alpha$. To have x not in $\bigcup_{\alpha\in\Lambda} A_\alpha$ we must have $x \notin A_\alpha$ for all $\alpha \in \Lambda$, or, equivalently, $x \in A_\alpha^C$ for all $\alpha \in \Lambda$. Now, if $x \in A_\alpha^C$ for all $\alpha \in \Lambda$ then $x \in \bigcap_{\alpha\in\Lambda} A_\alpha^C$, and so we have relationship 1.

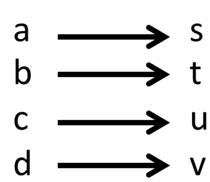
To prove relationship 2, let x be any member of $\bigcap_{\alpha\in\Lambda} A_\alpha^C$. This means that x is a member of A_α^C for all $\alpha \in \Lambda$, i.e., x is not in A_α for all $\alpha \in \Lambda$. But if x is not in A_α for all $\alpha \in \Lambda$, then x is also not in $\bigcup_{\alpha\in\Lambda} A_\alpha$, i.e., x is in $(\bigcup_{\alpha\in\Lambda} A_\alpha)^C$. Thus we have relationship 2.

Since we have both relationships 1 and 2, we can conclude that if Λ is a nonempty indexing set and $\mathcal{A} = \{A_\alpha | \alpha \in \Lambda\}$ is an indexed family of sets, then

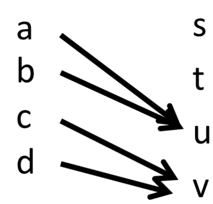
$$\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^C = \bigcap_{\alpha\in\Lambda} A_\alpha^C. \qquad \square$$

Problem 4. (Given $A = \{a, b, c, d\}$, $B = \{a, b, c\}$, and $C = \{s, t, u, v\}$, give arrow diagrams for functions as follows or explain why such functions cannot exist.)

A function $f : A \to C$ whose range is C:



A function $f : A \to C$ whose range is $\{u, v\}$:



A function $f : B \to C$ whose range is C. This function can't exist because B is smaller than C — since each member of B can only map to one member of C, it's not possible to hit all the members of C.

A function $f : A \to C$ with the property that for all x and y in A, if $x \neq y$ then $f(x) \neq f(y)$. The function shown at the beginning of this problem has this property.

A function $f : A \to \{s, t, u\}$ with the property that for all x and y in A, if $x \neq y$ then $f(x) \neq f(y)$. This is not possible because $\{s, t, u\}$ is smaller than A — anything that maps all members of A, as a function must, has to map at least 2 to the same member of $\{s, t, u\}$.

Problem 5. Define $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by

$$g(x) = (x, 2x). \quad (3)$$

Then we have ...

Proposition 3. The function g defined by Equation 3 is a homomorphism.

Proof. To prove that g is a homomorphism, we must prove that for all x and y in $\mathbb{Z}$, $g(x + y) = g(x) + g(y)$. So let x and y be integers. Then we see that

$$\begin{aligned} g(x+y) &= (x+y, 2(x+y)) \\ &= (x+y, 2x+2y) \\ &= (x, 2x) + (y, 2y) \\ &= g(x) + g(y) \end{aligned}$$

We have thus shown that $g(x + y) = g(x) + g(y)$, and therefore that g is a homomorphism. □