Math 239 Problem Set 10 Solution

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Here are my solutions to the questions in Problem Set 10.

Problem 1. (Are $A \cap B$ and $A - B$ disjoint? I think they are, so...)

Proposition 1. If A and B are subsets of universal set U, then $A \cap B$ is disjoint from $A - B$.

Proof. Let A and B be subsets of universal set U. We will prove that $A \cap B$ is disjoint from $A - B$ by contradiction. In other words, assume that the sets are not disjoint, so there is some element x that is a member of both. Since x is in $A \cap B$, x must be a member of B. But since x is also in $A - B$ it must not be a member of B. This is the contradiction, because x cannot both be in B and not be in B. We therefore conclude that $A \cap B$ must be disjoint from $A - B$. □

Problem 2. (Give an algebraic proof that $A - (A \cap B^C) = A \cap B$.)

Proposition 2. If A and B are subsets of some universal set U, then $A - (A \cap B^C) = A \cap B$.

Proof. Let A and B be subsets of universal set U. We will prove via a direct proof that $A - (A \cap B^C) = A \cap B$. Using laws of set algebra, we see that

$$\begin{aligned} A - (A \cap B^C) &= A \cap (A \cap B^C)^C \\ &= A \cap (A^C \cup (B^C)^C) \\ &= A \cap (A^C \cup B) \\ &= (A \cap A^C) \cup (A \cap B) \\ &= \emptyset \cup (A \cap B) \\ &= A \cap B \end{aligned}$$

□

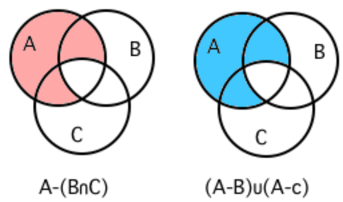
Notice that this proof used the "law" that $A \cap A^C = \emptyset$, which we technically haven't seen proven yet. So we should provide our own proof, as follows:

Lemma 1. If A is a subset of some universal set U, then $A \cap A^C = \emptyset$.

Proof. Let A be a subset of universal set U. We will prove by contradiction that $A \cap A^C = \emptyset$. In other words, assume that $A \cap A^C$ is not empty, so that there is some element x that is a member of $A \cap A^C$. Thus x must be in A, and x must also be in A^C. But the latter means that x is not in A, which is the contradiction: x cannot both be in A and not be in A. Therefore we conclude that if A is a subset of some universal set U, then $A \cap A^C = \emptyset$. □

Problem 3. (Use Venn diagrams to form a conjecture about the relationship between $A - (B \cap C)$ and $(A - B) \cup (A - C)$. Then prove that conjecture using both the choose-an-element method and set algebra.)

Here are the Venn diagrams:



They suggest the following:

Proposition 3. If A, B, and C are subsets of some universal set U, then $A - (B \cap C) = (A - B) \cup (A - C)$.

The first proof is as follows:

Proof. Let A, B, and C be subsets of some universal set U. We use the choose-an-element method to prove that $A - (B \cap C) = (A - B) \cup (A - C)$. Specifically, we show that x is in $A - (B \cap C)$ if and only if x is in $(A - B) \cup (A - C)$. So notice that

$$x \in A - (B \cap C) \text{ if and only if } x \in A \wedge x \notin (B \cap C)$$

Furthermore

$$x \in A \wedge x \notin (B \cap C) \text{ if and only if } x \in A \wedge (x \notin B \vee x \notin C)$$

and

$$x \in A \wedge (x \notin B \vee x \notin C) \text{ if and only if } (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$$

Moreover

$$(x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C) \text{ if and only if } (x \in A - B) \vee (x \in A - C)$$

Finally, observe that

$$(x \in A - B) \vee (x \in A - C) \text{ if and only if } x \in (A - B) \cup (A - C)$$

We have thus proven, through the above series of biconditionals, that if A, B, and C are subsets of some universal set U, then $A - (B \cap C) = (A - B) \cup (A - C)$. □

Here is the second proof:

Proof. Let A, B, and C be subsets of some universal set U. We use a direct proof based on set algebra to prove that $A - (B \cap C) = (A - B) \cup (A - C)$. In particular, algebraic laws tell us that

$$\begin{aligned} A - (B \cap C) &= A \cap (B \cap C)^C \\ &= A \cap (B^C \cup C^C) \\ &= (A \cap B^C) \cup (A \cap C^C) \\ &= (A - B) \cup (A - C) \end{aligned}$$

We have thus proven that if A, B, and C are subsets of some universal set U, then $A - (B \cap C) = (A - B) \cup (A - C)$. □

Problem 4. Let $f(n)$ be the function defined by

$$f(n) = \begin{cases} 1 \text{ if } n = 1, \\ 2 + f(n-1) \text{ if } n > 1. \end{cases} \tag{1}$$

We then have the following

Proposition 4. If $f(n)$ is as defined in Equation 1, then $f(n) = 2n - 1$.

Proof. We prove that $f(n) = 2n - 1$ by induction on n.

For the basis step, let $n = 1$. From the first case in Equation 1, $f(1) = 1$, and we also see that $2 \times 1 - 1 = 1$. This establishes the basis step for the induction.

For the induction step, we show that for all natural numbers $k \geq 1$, if $f(k) = 2k - 1$, then $f(k+1) = 2(k+1) - 1$. So assume that $f(k) = 2k - 1$. Since $k \geq 1$, $k+1 > 1$, and so $f(k+1)$ is defined by the second case in Equation 1. In other words,

$$\begin{aligned} f(k+1) &= 2 + f(k) \\ &= 2 + 2k - 1 \\ &= 2k + 2 - 1 \\ &= 2(k+1) - 1 \end{aligned}$$

This completes the inductive step.

Having established both a basis step and an inductive step, we conclude by the principle of mathematical induction that if $f(n)$ is defined by Equation 1, then $f(n) = 2n - 1$. □