Math 239 Exam 2 Solution

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Here are my solutions to the questions on hour exam 2.

Problem 1. (Prove that if a and b are integers and the product ab is not divisible by 9, then at least one of a and b is not divisible by 3.)

Proof. The proof is via the contrapositive, i.e., we prove that if a and b are integers and a and b are both divisible by 3, then ab is divisible by 9. Since a and b are divisible by 3, there exist integers m and n such that $a = 3m$ and $b = 3n$. The product ab can then be written as

$$\begin{aligned} ab &= (3m)(3n) \\ &= 9mn \end{aligned}$$

Since the integers are closed under multiplication, mn is an integer, and so we see that ab is divisible by 9. Since we have proven its contrapositive, we conclude that if a and b are integers and the product ab is not divisible by 9, then at least one of a and b is not divisible by 3. $\square$

Problem 2. (Prove that for all natural numbers n, 11^n is odd.)

Proof. The proof is by induction on n.

For the basis step, consider $n = 1$. $11^1 = 11$, which is odd. The basis step is thus established.

For the induction step, we show that for any natural number k, if 11^k is odd then 11^{k+1} is also odd. Assume that 11^k is odd, and note that $11^{k+1} = 11 \times 11^k$. Since 11 and 11^k are both odd, their product is also odd, establishing that 11^{k+1} is indeed odd.

Having established both a basis step and an induction step, we conclude by the principle of mathematical induction that 11^n is odd for all natural numbers n. $\square$

Problem 3. (Prove that no integer n has the property that $n \equiv 1 \pmod 2$ and $n \equiv 4 \pmod 6$.)

Proof. The proof is by contradiction. Assume that there is some integer n such that $n \equiv 1 \pmod 2$ and $n \equiv 4 \pmod 6$. Since $n \equiv 1 \pmod 2$ we know that

$n = 2a + 1$ for some integer a. This means that n is odd. Further, since $n \equiv 4 \pmod 6$ we see that for some integer b

$$\begin{aligned} n &= 6b + 4 \\ &= 2(3b + 2) \end{aligned}$$

which shows that n is even. We now have a contradiction, because no integer can be both odd and even. We thus conclude that indeed no integer n has the property that $n \equiv 1 \pmod 2$ and $n \equiv 4 \pmod 6$. □