Math 239 Problem Set 8 Solution

Problem 1. Is the set $\{4n-1|n \in \mathbb{N}\}$ inductive?

This set can be written as $\{3, 7, 11, \ldots\}$, showing two examples of natural numbers k for which $k+1$ is not in the set. The set is therefore not inductive.

Problem 2. Prove that

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Proof. The proof is by induction on n.

For the basis step, we let $n = 1$ and show that

$$\sum_{i=1}^{1} i^3 = \left[\frac{1 \times 2}{2}\right]^2$$

From the definition of summation

$$\begin{aligned}\sum_{i=1}^{1} i^3 &= 1 \\ &= 1^2 \\ &= \left[\frac{1 \times 2}{2}\right]^2\end{aligned}$$

For the inductive step, we assume that for some natural number $k \geq 1$

$$\sum_{i=1}^{k} i^3 = \left[\frac{k(k+1)}{2}\right]^2$$

and show that

$$\sum_{i=1}^{k+1} i^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2$$

Again using the definition of summation and algebra

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\
&= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)(k+1)^2}{4} \\
&= \frac{(k^2+4(k+1))(k+1)^2}{4} \\
&= \frac{(k^2+4k+4)(k+1)^2}{4} \\
&= \frac{(k+2)^2(k+1)^2}{4} \\
&= \left[\frac{(k+1)(k+2)}{2}\right]^2
\end{aligned}$$

In other words, we have shown that if

$$\sum_{i=1}^{k} i^3 = \left[\frac{k(k+1)}{2}\right]^2$$

then

$$\sum_{i=1}^{k+1} i^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2$$

This result, and the basis step, prove by the principle of mathematical induction that

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

for all natural numbers n. □

Problem 3. Formulate and prove a conjecture about what the n^{th} derivative of e^{ax} is

Proposition 1. For all natural numbers n,

$$\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$$

Proof. The proof is by induction on n.

For the basis step, we let $n = 1$ and show that

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

This result follows immediately from the chain rule and the fact that

$$\frac{d}{dx}(e^{u}) = e^{u}$$

For the inductive step, we assume that for some natural number $k \geq 1$

$$\frac{d^k}{dx^k}(e^{ax}) = a^k e^{ax}$$

and show that

$$\frac{d^{k+1}}{dx^{k+1}}(e^{ax}) = a^{k+1} e^{ax}$$

From the definition of the $(k+1)^{\text{th}}$ derivative, and noting that $a^k e^{ax}$ is differentiable, we have

$$\begin{aligned}
\frac{d^{k+1}}{dx^{k+1}}(e^{ax}) &= \frac{d}{dx}(\frac{d^k}{dx^k}(e^{ax})) \\
&= \frac{d}{dx}(a^k e^{ax}) \\
&= a \times a^k e^{ax}) \\
&= a^{k+1} e^{ax}
\end{aligned}$$

We have thus shown that if, for some natural number $k \geq 1$

$$\frac{d^k}{dx^k}(e^{ax}) = a^k e^{ax}$$

then

$$\frac{d^{k+1}}{dx^{k+1}}(e^{ax}) = a^{k+1} e^{ax}$$

This result, together with the basis step, establishes by the principle of mathematical induction that for all natural numbers n,

$$\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$$

□

Problem 4. Formulate and prove a proposition of the form "for all natural numbers $n \geq k$, there exist non-negative integers x and y such that $n = 4x+5y$," where k is a constant to be discovered by you.

Following hints in Sundstrom's Progress Check 4.10, I suspect that if I can find 4 consecutive natural numbers that can be written in the form $4x + 5y$, I will be able to prove that all larger naturals can also be written in that form. By trying successive natural numbers, I find that the first group of four that can be written in the desired form is 12 ($= 4 \times 3 + 5 \times 0$), 13 ($= 4 \times 2 + 5$), 14 ($= 4 + 5 \times 2$), and 15 ($= 4 \times 0 + 5 \times 3$)). I thus conjecture

Proposition 2. For all natural numbers $n \geq 12$, there exist non-negative integers x and y such that $n = 4x + 5y$.

Proof. The proof is by induction on n. In the following, let $P(n)$ be the predicate "there exist non-negative integers x and y such that $n = 4x + 5y$".

For the basis step, note as shown above that $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are all true.

For the inductive step, assume that for some natural number $k \geq 15$, P is true of all natural numbers m with $12 \leq m \leq k$, and show that $P(k+1)$ is also true. Note that $k + 1 = 4 + (k + 1) - 4$, and that $12 \leq (k + 1) - 4 \leq k$. Thus there exist integers x and y such that $(k + 1) - 4 = 4x + 5y$. Then

$$\begin{aligned} k+1 &= 4 + (k+1) - 4 \\ &= 4 + 4x + 5y \\ &= 4(x+1) + 5y \end{aligned}$$

Since the integers are closed under addition, $x + 1$ is an integer, and so there exist integers $z = x + 1$ and y such that $k + 1 = 4z + 5y$, i.e., $P(k + 1)$ is true.

This result and the basis step establish, by the second principle of mathematical induction, that for all natural numbers $n \geq 12$, there exist non-negative integers x and y such that $n = 4x + 5y$. $\square$