Math 239 Problem Set 6 Solution

Problem 1. Prove that for all integers a and b, if $\sqrt{ab} \neq \frac{a+b}{2}$ then $a \neq b$.

Proof. We prove the contrapositive, namely that if $a = b$ then $\sqrt{ab} = \frac{a+b}{2}$. Since $a = b$,

$$\begin{aligned} \sqrt{ab} &= \sqrt{a^2} \\ &= a \\ &= \frac{2a}{2} \\ &= \frac{a+a}{2} \\ &= \frac{a+b}{2} \end{aligned}$$

We have thus shown that for all integers a and b, if $a = b$ then $\sqrt{ab} = \frac{a+b}{2}$. The contrapositive of the conditional is also true, i.e., for all integers a and b, if $\sqrt{ab} \neq \frac{a+b}{2}$ then $a \neq b$, which is what we set out to prove. □

Problem 2. Prove that for all integers n, $6|n$ if and only if $2|n$ and $3|n$.

Proof. We prove each direction separately.

For the first part of the proof, we show that if $6|n$, then $2|n$ and $3|n$. Assume that 6 divides n. Then there is some integer a such that $n = 6a$. But since $6 = 3 \times 2$, this can be rewritten as $n = 3(2a)$ and $n = 2(3a)$. Since the integers are closed under multiplication, $2a$ and $3a$ are both integers, so $3|n$ and $2|n$. Thus we have shown that if $6|n$, then $2|n$ and $3|n$, which completes the first part of the proof.

For the second part of the proof, we show that if $2|n$ and $3|n$, then $6|n$. Assume that $2|n$ and $3|n$. Then there exist integers a and b such that $n = 2a$ and $n = 3b$. Since $n = 2a$ for an integer a, n is even. Furthermore $n = 3b$, and 3 is odd, so by the theorem proven in class on October 3 (for all integers a and b, if ab is even but a is odd, then b is even), b must be even. In other words there is some integer c such that $b = 2c$. Thus

$$\begin{aligned} n &= 3b \\ &= 3(2c) \\ &= 6c \end{aligned}$$

In other words, 6 divides n. We have thus shown that if $2|n$ and $3|n$, then $6|n$, completing the second part of the proof.

Since we have shown, for all integers n, both that if $6|n$ then $2|n$ and $3|n$, and that if $2|n$ and $3|n$ then $6|n$, we have shown that for all integers n, $6|n$ if and only if $2|n$ and $3|n$. $\square$

Problem 3. Generalize the proposition that for each real number x, $x + \sqrt{2}$ is irrational or $-x + \sqrt{2}$ is irrational to all irrational numbers.

Proposition 1. For all real numbers x and all irrational numbers a, $x + a$ is irrational or $-x + a$ is irrational.

Proof. The proof is by contradiction. Assume that there exist a real number x and an irrational number a such that $x+a$ is rational and $-x+a$ is rational. Since the rational numbers are closed under addition, the sum $x + a + (-x) + a = 2a$ must also be rational. But by Sundstrom's Proposition 3.19 (for all real numbers x and y where x is rational and not zero, and y is irrational, xy is irrational), $2a$ must be irrational, which is a contradiction. Thus there cannot exist a real number x and irrational number a such that $x + a$ is rational and $-x + a$ is rational, and so it must instead be true that for all real numbers x and all irrational numbers a, $x + a$ is irrational or $-x + a$ is irrational. $\square$

Problem 4. Prove that for all real numbers a and b, if $a \neq 0$ and $b \neq 0$ then $\sqrt{a^2 + b^2} \neq a + b$.

Proof. The proof is by contradiction. Assume that there exist real numbers a and b such that $a \neq 0$ and $b \neq 0$ and $\sqrt{a^2 + b^2} = a + b$. Then by algebra

$$\begin{aligned} a^2 + b^2 &= (a + b)^2 \\ &= a^2 + 2ab + b^2 \end{aligned}$$

But since $a \neq 0$ and $b \neq 0$, $2ab \neq 0$, and so $a^2 + 2ab + b^2$ cannot equal $a^2 + b^2$. Thus there must not exist real numbers a and b such that $a \neq 0$ and $b \neq 0$ and $\sqrt{a^2 + b^2} = a + b$; rather it must be true that for all real numbers a and b, if $a \neq 0$ and $b \neq 0$ then $\sqrt{a^2 + b^2} \neq a + b$. $\square$