Math 239 Problem Set 12 Solution

Problem 1. Given $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$, find and prove correct a simple expression for $f^n(x)$.

Proposition 1.1. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x+1$, then $f^n(x) = x+n$.

Proof. The proof is by induction on n.

For the basis step, let $n = 1$. From the definition of the f^n notation,

$$\begin{aligned} f^1(x) &= f(x) \\ &= x+1 \\ &= x+n \end{aligned}$$

We have thus completed the basis step, showing that $f^1(x) = x + 1$

For the induction step, we assume that for some natural number $k \geq 1$, $f^k(x) = x+k$. We will then show that $f^{k+1}(x) = x+k+1$. From the definition of the f^n notation,

$$\begin{aligned} f^{k+1}(x) &= (f \circ f^k)(x) \\ &= f(f^k(x)) \\ &= f(x+k) \\ &= x+k+1 \end{aligned}$$

We have thus established that for all natural numbers k, if $f^k(x) = x + k$ then $f^{k+1}(x) = x + k + 1$.

We have now proven by induction that if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x+1$, then $f^n(x) = x + n$. $\square$

Problem 2. Prove the following:

Proposition 2.1. If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection.

Proof. We prove that if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection by first noting that f^{-1} is a function because f is a bijection. We then show that f^{-1} is a bijection by showing that it is an injection and a surjection.

To show that f^{-1} is an injection, suppose that b_1 and b_2 are elements of B such that $f^{-1}(b_1) = f^{-1}(b_2) = a$. Then $f(a) = b_1$ and $f(a) = b_2$. Since f is a function, b_1 must equal b_2 in order for this equality to hold. Thus, whenever $f^{-1}(b_1) = f^{-1}(b_2)$, $b_1 = b_2$, and so f^{-1} is an injection.

To show that f^{-1} is a surjection, let a be any element of A. Since f is a function from A to B, there must be some b in B such that $f(a) = b$. But then $f^{-1}(b) = a$. We have thus shown that every element of A is the image under f^{-1} of some element of B, so f^{-1} is a surjection.

Since f^{-1} is an injection and f^{-1} is a surjection, we have shown that if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection. $\square$

Problem 3. Prove the following:

Proposition 3.1. If $f : S \to T$ is a function and $C \subseteq T$, then $f(f^{-1}(C)) \subseteq C$.

Proof. We prove that if $f : S \to T$ is a function and $C \subseteq T$, then $f(f^{-1}(C)) \subseteq C$ by showing that any x that is a member of $f(f^{-1}(C))$ is also a member of C. Let x be in $f(f^{-1}(C))$. Then there is a y in $f^{-1}(C)$ such that $x = f(y)$. But $f^{-1}(C)$ is by definition the set $\{y \in S | f(y) \in C\}$, so $f(y) = x \in C$. Thus any x that is a member of $f(f^{-1}(C))$ is also a member of C, proving that $f(f^{-1}(C)) \subseteq C$. $\square$

Problem 4. Given that for any real numbers $x > 0$ and y there exists a natural number n such that $nx > y$ (Fact 1), and that for any real number x there is an integer m such that $x \leq m$ but $x \not\leq m - 1$ (Fact 2), prove the following propositions and theorem:

Proposition 4.1. For any real numbers x and y such that $x < y$, there is a natural number n such that $ny - nx > 1$.

Proof. First note that since $1 > 0$, a corollary to Fact 1 is that for any real number $z > 0$, there exists a natural number n such that $nz > 1$. Then let $z = y - x$. Since $x < y$, $z > 0$, and so by the corollary to Fact 1 there exists a natural number n such that $nz > 1$. But since $z = y - x$,

$$\begin{aligned} nz &= n(y - x) \\ &= ny - nx \end{aligned}$$

Thus $ny - nx > 1$. We have therefore proven that for any real numbers x and y such that $x < y$, there is a natural number n such that $ny - nx > 1$. $\square$

Proposition 4.2. If x and y are real numbers such that $x < y$ and n is a natural number such that $ny - nx > 1$, then there is an integer q such that $nx < q < ny$.

Proof. Let x and y be real numbers such that $x < y$, and let n be a natural number such that $ny - nx > 1$. From $ny - nx > 1$, we see that $ny > 1 + nx$, and so in turn $ny - 1 > nx$, or equivalently,

$$nx < ny - 1 \tag{1}$$

Now, since the real numbers are closed under multiplication and every natural number is also real, ny is a real number, and so by Fact 2 there exists an integer

m such that $ny \leq m$ but $ny \not\leq m - 1$. Realizing that "not less than or equal" is equivalent to "greater than," we can rewrite these last relations as

$$m - 1 < ny \leq m \tag{2}$$

Subtracting 1 from both sides of the right-hand inequality in Equation 2 gives

$$ny - 1 \leq m - 1 \tag{3}$$

Combining Equations 1, 2, and 3 we see that

$$nx < ny - 1 \leq m - 1 < ny$$

or

$$nx < m - 1 < ny$$

Now let q be $m - 1$. Since m is an integer and the integers are closed under subtraction, this q is an integer, and so we have established that there exists an integer q such that $nx < q < ny$. Thus, if x and y are real numbers such that $x < y$ and n is a natural number such that $ny - nx > 1$, then there is an integer q such that $nx < q < ny$. □

Theorem 4.3. For any real numbers x and y such that $x < y$, there is a rational number s such that $x < s < y$.

Proof. Since $x < y$, Propositions 4.1 and 4.2 say that there is some natural number n and some integer q such that

$$nx < q < ny$$

Now divide all three parts by n to get

$$x < \frac{q}{n} < y$$

Since n is a natural number, $n \neq 0$, and so $\frac{q}{n}$ is a rational number. Call this rational number s, and we have shown that for any real numbers x and y such that $x < y$, there is a rational number s such that $x < s < y$. □