Math 239 Problem Set 10 Solution

Problem 1. Letting Λ be an indexing set, $\mathcal{A} = \{A_\alpha | \alpha \in \Lambda\}$ be an indexed family of sets, and C be a set such that $A_\alpha \subseteq C$ for all $\alpha \in \Lambda$, prove that

$$\bigcup_{\alpha\in\Lambda} A_\alpha \subseteq C$$

Proof. We prove that $\bigcup_{\alpha\in\Lambda} A_\alpha$ is a subset of C by showing that every element of $\bigcup_{\alpha\in\Lambda} A_\alpha$ is also an element of C. Let x be any element of $\bigcup_{\alpha\in\Lambda} A_\alpha$. Then by the definition of the union of an indexed family of sets, there is some A_i such that $x \in A_i$. Now since A_i is a subset of C, x is also a member of C. Thus we have shown that any $x \in \bigcup_{\alpha\in\Lambda} A_\alpha$ is also in C, so $\bigcup_{\alpha\in\Lambda} A_\alpha \subseteq C$. $\square$

Problem 2. Determine whether functions f and g are equal, where $f : \mathbb{Z}_6 \to \mathbb{Z}_6$ is defined by $f(x) = (x^2 + 4) \pmod 6$ and $g : \mathbb{Z}_6 \to \mathbb{Z}_6$ is defined by $g(x) = (x + 1)(x + 4) \pmod 6$.

The first two parts of the question ask us to calculate the values of f and g, as follows:

x	$f(x)$	$g(x)$
0	4	4
1	5	4
2	2	0
3	1	4
4	2	4
5	5	0

Based on this table, f and g are not equal, because even though their domains and codomains are equal, there are values of x (e.g., $x = 1$) for which $f(x) \neq g(x)$.

Problem 3. Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$$f(n) = \frac{1 + (-1)^n(2n - 1)}{4}$$

and determine whether f is an injection and whether it is a surjection. Give a proof or counterexample for each conclusion.

In analyzing f, it is convenient to notice that for all natural numbers n, $2n - 1 \geq 1$, and therefore $f(n)$ can be less than or equal to 0 only if n is odd. Furthermore, since -1^n is negative for all odd naturals n, $f(n) \leq 0$ whenever

n is odd. Thus $f(n) \leq 0$ if and only if n is odd, which in turn implies that $f(n) > 0$ if and only if n is even.

Proposition 3.1. f is an injection.

Proof. We prove that f is an injection by showing that if $f(n) = f(m)$ then $n = m$. Assume that n and m are natural numbers such that $f(n) = f(m)$, and consider two cases: n is odd, or n is even.

For the first case, when n is odd, $f(n) \leq 0$. Since $f(n) = f(m)$, $f(m)$ must also be less than or equal to 0, and so m is also odd. Then $f(n) = f(m)$ means

$$\frac{1-(2n-1)}{4} = \frac{1-(2m-1)}{4}$$

and so $n = m$.

For the second case, when n is an even natural number, $f(n)$ is positive. $f(m)$ must also be positive, implying that m is even. Then $f(n) = f(m)$ means

$$\frac{1+(2n-1)}{4} = \frac{1+(2m-1)}{4}$$

and so $n = m$.

We have shown that in both cases, $f(n) = f(m)$ implies that $n = m$, and so f is an injection. □

Proposition 3.2. f is a surjection.

Proof. We prove that f is a surjection by showing that for every integer x there is a natural number n such that $x = f(n)$. The proof is in two cases, namely $x \leq 0$ and $x > 0$.

For the first case, we let $x \leq 0$ be an integer, and find an odd natural number n such $f(n) = x$. Finding an odd natural number n such that $f(n) = x$ is equivalent to finding an integer $m \geq 0$ such that $f(2m+1) = x$. By basic algebra

$$\begin{aligned} x &= f(2m+1) \\ &= \frac{1+(-1)^{2m+1}(2(2m+1)-1)}{4} \\ &= \frac{1-(4m+2-1)}{4} \\ &= \frac{-4m}{4} \\ &= -m \end{aligned}$$

Thus any integer $x \leq 0$ is equal to $f(-2x+1)$; since $-2x+1$ is an integer greater than or equal to 1, it is also a natural number and so in the domain of f.

For the second case, we let $x > 0$ be an integer and find an even natural number n such that $f(n) = x$. Finding an even natural number n such that

$f(n) = x$ is equivalent to finding any natural number m such that $f(2m) = x$. By basic algebra

$$\begin{aligned} x &= f(2m) \\ &= \frac{1 + (-1)^{2m}(2(2m) - 1)}{4} \\ &= \frac{1 + (2(2m) - 1)}{4} \\ &= \frac{1 + (4m - 1)}{4} \\ &= \frac{4m}{4} \\ &= m \end{aligned}$$

In other words, any integer $x > 0$ is equal to $f(2x)$; $2x$ is a natural number and so in the domain of f.

We have now shown that every integer can be computed as f of some natural number, and so f is a surjection. □

Problem 4. Given that for any real numbers $x > 0$ and y there exists a natural number n such that $nx > y$ (Fact 1), and that for any real number x there is an integer m such that $x \leq m$ but $x \not\leq m - 1$ (Fact 2), prove the following two propositions:

Proposition 4.1. For every real number $x > 0$, there exists a natural number n such that $0 < \frac{1}{n} < x$.

Proof. It is convenient to rename the variable x in the proposition to avoid confusion with x in Fact 1. Specifically, we show that for every real number $z > 0$, there exists a natural number n such that $0 < \frac{1}{n} < z$. Let z be a real number greater than 0. Then $\frac{1}{z}$ is also a real number greater than 0, call it y. By Fact 1 with $x = 1$, there exists a natural number n such that $n > y$. Since $n > y$ and $y > 0$, $\frac{1}{n} < \frac{1}{y} = z$. Furthermore, since n is a natural number, $n > 0$ and so $\frac{1}{n}$ is also greater than 0. Thus we have shown that there exists a natural number n such that $0 < \frac{1}{n} < z$, as claimed by the proposition. □

Proposition 4.2. For any real numbers x and y such that $x < y$, there is a natural number n such that $x < x + \frac{1}{n} < y$.

Proof. We show that for any real numbers x and y such that $x < y$, there is a natural number n such that $x < x + \frac{1}{n} < y$. Let x and y be real numbers with $x < y$, and let $z = y - x$. Since $x < y$, $z > 0$, and so by Proposition 4.1, there exists a natural number n such that $0 < \frac{1}{n} < z$. Adding x to each part of this inequality yields $x < x + \frac{1}{n} < x + z$. But $x + z = x + (y - x) = y$, and so we have $x < x + \frac{1}{n} < y$, as claimed by the proposition. □