Math 239 Problem Set 10 Solution

Problem 1. Prove that $(A \cap B)^C = A^C \cup B^C$.

Proof. Let A and B be subsets of some universal set U (so that their complements, and the complement of their intersection, are well-defined). We will then show that $(A \cap B)^C = A^C \cup B^C$ by showing that for all elements x of U, the statement $x \in (A \cap B)^C$ is equivalent to $x \in A^C \cup B^C$, and thus that x is a member of one of the sets if and only if it is a member of the other.

$$\begin{aligned} x \in (A \cap B)^C &\equiv \neg(x \in A \cap B) \\ &\equiv \neg(x \in A \wedge x \in B) \\ &\equiv x \notin A \vee x \notin B \\ &\equiv (x \in A^C) \vee (x \in B^C) \\ &\equiv x \in A^C \cup B^C \end{aligned}$$

Thus $x \in (A \cap B)^C$ if and only if $x \in A^C \cup B^C$, and so in turn $(A \cap B)^C = A^C \cup B^C$. □

Problem 2. Form and give two proofs of a conjecture about the relationship between the sets $A - (B \cap C)$ and $(A - B) \cup (A - C)$.

The conjecture is that $A - (B \cap C) = (A - B) \cup (A - C)$, where A, B, and C are subsets of some universal set U.

The first proof is based on the Boolean definitions of set difference, intersection, and union:

Proof. Let x be any member of U. We show that$A-(B\cap C) = (A-B)\cup(A-C)$ by showing that x is a member of $A - (B \cap C)$ if and only if it is a member of $(A - B) \cup (A - C)$. We show this, in turn, by showing that the statement $x \in A - (B \cap C)$ is logically equivalent to the statement $x \in (A - B) \cup (A - C)$.

$$\begin{aligned} x \in A - (B \cap C) &\equiv x \in A \wedge \neg(x \in B \cap C) \\ &\equiv x \in A \wedge \neg(x \in B \wedge x \in C) \\ &\equiv x \in A \wedge (x \notin B \vee x \notin C) \\ &\equiv (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C) \\ &\equiv x \in (A - B) \vee x \in (A - C) \\ &\equiv x \in (A - B) \cup (A - C) \end{aligned}$$

We have thus shown that x is a member of $A-(B\cap C)$ if and only if it is a member of $(A-B)\cup(A-C)$, which in turn proves that $A-(B\cap C) = (A-B)\cup(A-C)$. $\square$

The second proof uses algebraic rules for sets:

Proof. We use the rule that $X - Y = X \cap Y^C$, one of De Morgan's laws, and the distributive property of intersection over union to prove that $A - (B \cap C) = (A - B) \cup (A - C)$:

$$\begin{aligned} A - (B \cap C) &= A \cap (B \cap C)^C \\ &= A \cap (B^C \cup C^C) \\ &= (A \cap B^C) \cup (A \cap C^C) \\ &= (A - B) \cup (A - C) \end{aligned}$$

Thus $A - (B \cap C) = (A - B) \cup (A - C)$. $\square$

Problem 3. Find the Cartesian product of $\{-1, 0, 1\}$ and $\{a, b\}$.

By forming all ordered pairs whose first member comes from $\{-1, 0, 1\}$ and whose second member comes from $\{a, b\}$ we see that $\{-1, 0, 1\} \times \{a, b\} = \{(-1, a), (-1, b), (0, a), (0, b), (1, a), (1, b)\}$.

Problem 4. Prove that $A \times B = B \times A$ if and only if $A = B$.

Proof. We prove each direction of the biconditional separately.

First, we show that if $A \times B = B \times A$, then $A = B$. To do this, we assume that $A \times B = B \times A$ and show that every x in A is also in B, and that every y in B is also in A. Let x be any member of A, and y any member of B. Then $(x, y) \in A \times B$. Since $A \times B = B \times A$, (x, y) is also in $B \times A$, and so $x \in B$ and $y \in A$. We have thus shown that every element of A is an element of B, and every element of B is an element of A, and so $A = B$.

For the second direction, we show that if $A = B$ then $A \times B = B \times A$. We do this by substituting equals for equals. First, by substituting A for B we see that

$$A \times B = A \times A \tag{1}$$

Then substituting B for the first A

$$A \times A = B \times A \tag{2}$$

Combining (1) and (2) gives

$$A \times B = B \times A$$

We have now shown that if $A \times B = B \times A$, then $A = B$ and that if $A = B$ then $A \times B = B \times A$, so $A \times B = B \times A$ if and only if $A = B$. $\square$