Example Induction Proof

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Theorem 1. For all natural numbers n, every set with n elements has exactly 2^n distinct subsets.

Proof. The proof is by induction on n.

The basis case is $n = 1$. All sets with 1 element are of the form $\{a\}$ for some value a. Such a set has subsets $\{a\}$ and $\emptyset$; there are $2 = 2^1$ of these subsets, and they are distinct from each other. Thus we have shown that all sets with 1 element have exactly 2^1 distinct subsets.

For the induction step, assume that all sets of k elements have exactly 2^k distinct subsets, for some $k \geq 1$, and show that all sets of $k + 1$ elements have exactly 2^{k+1} distinct subsets. Every set S of $k + 1$ elements is of the form $\{a_1, a_2, \ldots, a_k, ak + 1\}$ where the a_i are any distinct values. Any subset of S either does or does not contain a_{k+1}. All the subsets that do not contain a_{k+1} are subsets of $\{a_1, a_2, \ldots, a_k\}$; because this is a set of k elements, there are 2^k such subsets, each distinct from the others. Each subset that does contain a_{k+1} can be constructed by adding a_{k+1} to one of the subsets that does not contain it. There are therefore 2^k subsets that do contain a_{k+1}, and they are distinct from each other because the elements other than a_{k+1} distinguish them, and distinct from all the subsets that do not contain a_{k+1} because a_{k+1} itself distinguishes them. We have thus identified a total of $2^k + 2^k = 2^{k+1}$ subsets, each distinct from the others. We have also determined that there are no other subsets of S. Thus we have shown that if all sets of k elements have exactly 2^k distinct subsets, for some $k \geq 1$, then all sets of $k+1$ elements have exactly 2^{k+1} distinct subsets.

We have thus shown, by the Principle of Induction, that for all natural numbers n, every set with n elements has exactly 2^n distinct subsets. $\square$