Solutions to Exam 2 Questions

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Question 1. Prove that $5 - \sqrt{2}$ is irrational.

Proof. The proof is by contradiction. Assume that $5 - \sqrt{2}$ is rational, and so there exist integers a and b with $b \neq 0$ such that

$$5 - \sqrt{2} = \frac{a}{b}$$

Now notice that $\sqrt{2} = 5 - (5 - \sqrt{2})$, but

$$\begin{aligned} 5 - (5 - \sqrt{2}) &= 5 - \frac{a}{b} \\ &= \frac{5b - a}{b} \end{aligned}$$

which is rational because $5b - a$ and b are both integers and $b \neq 0$. But we already know that $\sqrt{2}$ is irrational, and no number can be both rational and irrational. □

Question 2. Prove that for all natural numbers n,

$$\sum_{i=1}^{n} \sqrt[3]{i} \geq n$$

Proof. The proof is by induction on n.

The basis case is $n = 1$. In this case the sum is equal to 1, which is greater than or equal to 1.

For the inductive step, assume that k is a natural number such that

$$\sum_{i=1}^{k} \sqrt[3]{i} \geq k$$

and show that

$$\sum_{i=1}^{k+1} \sqrt[3]{i} \geq k + 1$$

This sum can be written as

$$\sum_{i=1}^{k+1} \sqrt[3]{i} = \sum_{i=1}^{k} \sqrt[3]{i} + \sqrt[3]{k+1}$$

The first term on the right is greater than or equal to k by the induction hypothesis, and the second term is greater than 1 because $k+1>1$. Thus

$$\sum_{i=1}^{k+1} \sqrt[3]{i} \geq k+1$$

This completes the inductive step.

We have now shown by induction that for all natural numbers n,

$$\sum_{i=1}^{n} \sqrt[3]{i} \geq n$$

□

Question 3. Let $B(x)$ be defined as

$$B(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 2-x & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

and prove that for all real numbers x, $0 \leq B(x) \leq 1$.

Proof. The proof is in three cases, namely $0 \leq x < 1$, $1 \leq x \leq 2$, and $x < 0$ or $x > 2$.

For the first case, $B(x) = x$ and so, since $0 \leq x < 1$, $0 \leq B(x) \leq 1$.

For the second case, $B(x) = 2-x$. Since $B(x)$ is linear in this case, $1 \leq x \leq 2$ implies that $0 \leq 2-x = B(x) \leq 1$.

For the third case, $B(x) = 0$, and $0 \leq 0 \leq 1$.

We have thus shown that in all possible cases, $0 \leq B(x) \leq 1$ for any real number x. □