Math 223 01
Prof. Doug Baldwin

Problem Set 9 — Multivariable Functions

Complete by **Monday, March 23**
Grade by **Wednesday, March 25**

Purpose

This problem set develops your understanding of multivariable functions, their limits, and some basic ideas about their derivatives. By the time you finish this problem set, I expect that you will be able to

- Calculate values of multivariable functions
- Plot functions of 2 variables with Mathematica
- Find limits of multivariable functions
- Show, when appropriate, that limits of multivariable functions don't exist
- Differentiate multivariable functions.

Background

This exercise is based on sections 13.1, 13.2, and the beginning of section 13.3 in our textbook. We covered these sections in classes between February 28 and March 5.

Activity

Solve the following problems.

Question 1. (Inspired by exercise 4 in section 13.1E of our textbook.)

An oxygen tank is formed from a right circular cylinder of radius r and height h, with hemispheres of radius r at the ends of the cylinder.

Part A. Express the tank's volume as a function of r and h (background: the volume of a cylinder of radius r and height h is $V = \pi r^2 h$; the volume of a hemisphere of radius r is $V = \frac{2}{3}\pi r^3$). Then use your function to find the volume of a tank with radius 2 and height 5, and of a tank with radius 1 and height 10.

Solution:

$$\begin{aligned} V(r,h) &= \pi r^2 h + 2(\frac{2}{3}\pi r^3) \\ &= \pi r^2 h + \frac{4}{3}\pi r^3 \end{aligned}$$

Using this function,

$$\begin{aligned}
V(2,5) &= \pi \times 2^2 \times 5 + \frac{4}{3} \times \pi \times 2^3) \\
&= 20\pi + \frac{32}{3}\pi \\
&= \frac{92\pi}{3} \\
V(1,10) &= \pi \times 1^2 \times 10 + \frac{4}{3} \times \pi \times 1^3) \\
&= 10\pi + \frac{4}{3}\pi \\
&= \frac{34\pi}{3}
\end{aligned}$$

Part B. Use Mathematica to plot your volume function over the region $0 \leq r \leq 5, 0 \leq h \leq 5$.

Part C. The Geneseo Oxygen Tank company makes a standard tank of radius 3 inches and height 10 inches. If they want to change these dimensions slightly in order to hold more oxygen in their standard tank, will they get more "bang for the buck" by increasing the radius or by increasing the height? In other words, does volume change faster with changes in radius or with changes in height when $r = 3$ and $h = 10$?

Solution:

$$\begin{aligned}
\frac{\partial V}{\partial r} &= 2\pi r h + 4\pi r^2 \\
\frac{\partial V}{\partial h} &= \pi r^2
\end{aligned}$$

Thus, at $(3,10)$, $\frac{\partial V}{\partial r} = 2\pi \times 3 \times 10 + 4\pi \times 3^2 = 96\pi$, and $\frac{\partial V}{\partial h} = \pi \times 3^2 = 9\pi$. The rate of change with respect to radius is much larger.

Question 2. (Exercise 18 in section 13.2E of our textbook.)

Determine whether

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$$

exists, and if so what it is.

Solution:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} &= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{(\sqrt{x^2+y^2+1}-1)(\sqrt{x^2+y^2+1}+1)} \\
&= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2+1-1} \\
&= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2} \\
&= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+1}+1 \\
&= \sqrt{0^2+0^2+1}+1 \\
&= 2
\end{aligned}$$

Question 3. (Inspired by exercise 28 in section 13.2E of our textbook.)

Part A. Show that

$$\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$$

does not exist.

Solution: Consider approaching $(0,0)$ along the path $x=0$:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2} &= \lim_{y\to 0} \frac{y^3}{y^2} \\
&= \lim_{y\to 0} y \\
&= 0
\end{aligned}$$

Now consider approaching along the path $x=y$:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2} &= \lim_{y\to 0} \frac{y^2+y^3}{y^2+y^2} \\
&= \lim_{y\to 0} \frac{y^2(1+y)}{2y^2} \\
&= \lim_{y\to 0} \frac{1+y}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Since the limits along different paths are different, the overall limit does not exist.

Part B. Use Mathematica to plot the function from Part A near the origin. Be prepared during grading to identify the feature(s) of the plot that correspond to the non-existence of the limit.

Question 4. Intuitively, a single-variable function, $f(x)$, is said to have a limit of $+\infty$ at a if you can make $f(x)$ arbitrarily large by picking values for x sufficiently close to a. Formally, the idea is that for every number M, there exists a positive number δ such that $f(x) > M$ whenever $a-\delta < x < a+\delta$.

Extend the intuitive and formal definitions of a (positive) infinite limit to two-variable functions. Then use your extended formal definition to show that

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = \infty$$

Solution: Intuitively, $f(x,y)$ has a limit of $+\infty$ at (a,b) if you can make $f(x,y)$ arbitrarily large by picking values for x and y sufficiently close to (a,b). Formally,

$$\lim_{(x,y)\to(a,b)} f(x,y) = +\infty$$

if, for every real number M, there exists a real number δ such that $f(x,y) > M$ whenever

$$\sqrt{(x-a)^2+(y-b)^2} < \delta.$$

Applying this to $f(x,y) = \frac{1}{x^2+y^2}$, let M be any real number. Then

$$\begin{aligned} \frac{1}{x^2+y^2} > M \quad &\to \quad \frac{1}{M} > x^2+y^2 \\ &\to \quad x^2+y^2 < \frac{1}{M} \\ &\to \quad \sqrt{x^2+y^2} < \frac{1}{\sqrt{M}} \end{aligned}$$

So for any M, $\delta = \frac{1}{\sqrt{M}}$ satisfies the definition of infinite limit.

Question 5. Find all the (first) partial derivatives of each of the following functions:

1. $f(x,y) = \sin(xy) - xy$
2. $g(x,y) = x\ln(x+y)$
3. $h(x,y,z,w) = \frac{x^2-y^2}{z^2+w^2}$

Solution:

$$\frac{\partial f}{\partial x} = y\cos(xy) - y \tag{1}$$

$$\frac{\partial f}{\partial y} = x\cos(xy) - x \tag{2}$$

$$\frac{\partial g}{\partial x} = \ln(x+y) + x\frac{1}{x+y} \tag{3}$$

$$= \ln(x+y) + \frac{x}{x+y} \tag{4}$$

$$\frac{\partial g}{\partial y} = \frac{x}{x+y} \tag{5}$$

$$\begin{aligned}
\frac{\partial h}{\partial x} &= \frac{2x}{z^2 + w^2} && (6)\\
\frac{\partial h}{\partial y} &= \frac{-2y}{z^2 + w^2} && (7)\\
\frac{\partial h}{\partial z} &= (x^2 - y^2)\frac{-2z}{(z^2 + w^2)^2} && (8)\\
&= \frac{-2z(x^2 - y^2)}{(z^2 + w^2)^2} && (9)\\
\frac{\partial h}{\partial w} &= (x^2 - y^2)\frac{-2w}{(z^2 + w^2)^2} && (10)\\
&= \frac{-2w(x^2 - y^2)}{(z^2 + w^2)^2} && (11)
\end{aligned}$$

Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the "Grade By" date above. If you worked in a group on this assignment and the group has one collective solution, the whole group can schedule a single meeting with me.