Math 223 01
Prof. Doug Baldwin

Problem Set 14 — Conservative Vector Fields and Green's Theorem

Complete by **Sunday, May 10**
Grade by **Thursday, May 14**

Purpose

This problem set reinforces your understanding of conservative vector fields and of Green's Theorem. By the time you finish this problem set, I expect that you will be able to...

- Reason about conservative vector fields
- Evaluate line integrals using the fundamental theorem for line integrals
- Find potential functions for conservative vector fields
- Use Green's Theorem to evaluate line integrals.

Background

This problem set is mainly based on sections 15.3 and 15.4 of our textbook, although 15.1 provides an introduction to conservative vector fields. We covered this material in classes between May 1 and 6.

Activity

Solve the following problems.

Question 1. Phineas Phoole is studying the conservative vector field

$$\mathbf{F}(x,y) = \left\langle \frac{2}{y} + 2xy, \frac{-2x}{y^2} + x^2 \right\rangle$$

which has potential function

$$f(x,y) = \frac{2x}{y} + yx^2.$$

Phineas notices that if he integrates $\mathbf{F}$ along the path $\mathbf{r}(t) = \langle t, 1\rangle, 0 \leq t \leq 1$ he gets exactly the same value as when he integrates $\mathbf{F}$ along the path $s(t) = \langle t, 2-t\rangle, 0 \leq t \leq 1$. Both of these paths end at $\mathbf{r}(1) = \mathbf{s}(1) = (1,1)$. Phineas concludes from this observation that he has discovered an "ultraconservative" field, in which not only are line integrals path independent, but they are also starting-point independent: a line integral along any path $\mathbf{u}(t), a \leq t \leq b$ is simply equal to $f(\mathbf{u}(b))$.

Part A. Check Phineas's calculations and verify that

$$\int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 \mathbf{F}(\mathbf{s}(t)) \cdot \mathbf{s}'(t)\, dt$$

Part B. Show that Phineas is wrong about the "ultraconservative" field (there actually is no such thing), i.e., that there are integrals of $\mathbf{F}$ that end at point $(1, 1)$ but do not equal the value you found in Part **??**.

Question 2. Suppose $\mathbf{F}$ and $\mathbf{G}$ are any two conservative vector fields. Show that ...

1. Vector field $\mathbf{F} + \mathbf{G}$ is conservative
2. Vector field $k\mathbf{F}$, where k is any scalar, is conservative.

Hint: Use the fact that conservative vector fields are gradient fields.

Question 3. Is the vector field

$$\mathbf{F}(x, y, z, w) = \left\langle \frac{1}{zw}, \frac{-1}{zw}, \frac{y-x}{z^2w}, \frac{y-x}{zw^2} \right\rangle$$

conservative? If so, find a potential function for it.

Question 4. For each of the following vector fields and paths, calculate the circulation line integral of the field along the path, using as many of the following methods as apply to that integral:

1. Direct evaluation of the integral via the $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ formula
2. Using the fundamental theorem for line integrals or related properties of line integrals in conservative vector fields
3. By converting the line integral to a double integral via Green's Theorem.

Part A. Field $\mathbf{F}(x, y) = \langle x, xy^2 \rangle$; the path is the triangle from $(0, 0)$ to $(1, 0)$ to $(1, 1)$ and back to $(0, 0)$, traversed in that order.

Part B. Field $\mathbf{F}(x, y) = \langle 2x + y, 2y + x \rangle$; the path is the part of the curve $y = x^2$ starting at point $(0, 0)$ and ending at $(2, 4)$.

Part C. Field $\mathbf{F}(x, y) = \langle y, x \rangle$; the path is the circle of radius 2 centered at point $(1, 2)$, traversed counterclockwise.

Follow-Up

I will grade this exercise in a video chat, or alternative meeting, with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please have a written solution to the exercise available at your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting half an hour long, and schedule it to finish before the end of the "Grade By" date above. You must meet individually with me, even if you worked in a group on this problem set.

To "attend" your meeting, simply go to our course room in Canvas at the time you signed up for, and I will meet you there. I will set the room up so that you can share files from your computer with me.