Math 223 01
Prof. Doug Baldwin

Problem Set 13 — Line Integrals and Vector Fields

Complete by **Sunday, May 3**
Grade by **Wednesday, May 6**

Purpose

This problem set reinforces your understanding of vector fields and line integrals (of both scalar and vector functions). By the time you finish this problem set, I expect that you will be able to

- Solve problems involving scalar line integrals
- Solve problems involving vector line integrals
- Calculate values of vector fields at points
- Plot vector fields with Mathematica.

Background

This problem set is based on sections 15.1 and 15.2 of our textbook. We covered, or will cover, that material in classes between April 23 and 29. I expect to talk about plotting vector fields in Mathematica on April 24.

Activity

Solve the following problems.

Question 1. Find the value of

$$\int_C xy^4 \, ds$$

where C is the right half of the circle $x^2 + y^2 = 16$ and is traversed clockwise.

Question 2. Find the value of

$$\int_C x + \sqrt{y} - z^2 \, ds$$

where C is the 2-part path that follows a straight line from point $(0, 0, 0)$ to point $(1, 0, 0)$, and then from there to point $(1, 1, 1)$.

Question 3. Consider the vector field

$$\mathbf{F}(x, y, z) = \langle xy, yz, xz \rangle$$

Part A. What is the value of this field at point $(1, 2, 3)$? How about at point $(0, 0, 0)$?

Part B. Use Mathematica to plot this field over the region $-3 \leq x \leq 3$, $-3 \leq y \leq 3$, $-3 \leq z \leq 3$.

Question 4. Imagine a large flat-bottom kitchen sink with a drain in the center. Water flows in a thin sheet over the bottom of this sink towards the drain. Because the water flows in a thin sheet, it can be thought of as 2 dimensional, and its velocity can be described by the vector field

$$\mathbf{v}(x, y) = \left\langle \frac{-x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right\rangle$$

This equation is relative to a coordinate system whose origin is at the center of the drain.

Part A. Show that the flux of this field across any circle around the drain is constant, i.e., that the flux of $\mathbf{v}(x, y)$ across circle C does not depend on the radius of C.

Part B. Physically, the flux of water across a circle around the drain is the volume of water crossing the circle per unit of time. If that volume per unit time remains constant as the circles get smaller, then the speed at which the water flows must increase (since the same volume of water is crossing a shorter circumference in the same time, and so the volume of water per unit circumference per unit time has to increase). Show that the speed represented by $\mathbf{v}(x, y)$ does indeed increase as distance from the origin decreases.

Part C. Use Mathematica to plot $\mathbf{v}(x, y)$ near the origin to check that it behaves as implied by the context of the question and Part B, in particular that the flow is everywhere towards the origin (the drain), and the magnitude of the field is larger near the origin than farther away from it.

Hint: When I tried to plot $\mathbf{v}(x, y)$ in any region that had the origin at the center, I got very unhelpful plots, probably because Mathematica found an infinitely long vector at $(0, 0)$ and couldn't find a good way to draw it and the finite vectors near it. Plotting over a slightly asymmetrical region that doesn't have the origin at the center (for example, $-0.9 \leq x \leq 1, -0.9 \leq y \leq 1$) helped with this problem. You may have similar experiences.

Part D. Show that the circulation of $\mathbf{v}(x, y)$ around any circle around the drain is 0, regardless of the circle's radius.

,

Follow-Up

I will grade this exercise in a video chat, or alternative meeting, with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please have a written solution to the exercise available at your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting half an hour long, and schedule it to finish before the end of the "Grade By" date above. You must meet individually with me, even if you worked in a group on this problem set.

To "attend" your meeting, simply go to our course room in Canvas at the time you signed up for, and I will meet you there. I will set the room up so that you can share files from your computer with me.