Problem Set 10 — Line Integrals

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Math 223 04

Complete By Tuesday, May 1
Grade By Tuesday, May 8

Purpose

This problem set develops your ability to evaluate and reason about line integrals and vector fields.

Background

This exercise is based on material from chapter 6 of our textbook. We discussed (or will discuss) this material in classes between April 11 and April 26.

Activity

Solve the following problems.

Problem 1. Spacey the space traveler is collecting genesium gas, a rare and incredibly valuable substance without which galactic civilization would not exist. Specifically, Spacey is flying through an interstellar cloud of the gas gathering it up. Relative to a coordinate system whose origin is at the center of cloud, the density of the genesium (in grams per cubic kilometer) is given by

$$\rho(x, y, z) = \frac{1}{x^2 + y^2 + z^2}$$

and Spacey's path through the cloud is given parametrically as

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

for $-1 \leq t \leq 1$. All positions and distances are in kilometers. Spacey's genesium collector is set up so that when flying distance ds kilometers through genesium of density ρ grams per cubic kilometer, it collects ρds grams of the gas. How much genesium does Spacey collect?

Problem 2. Lots of physical forces obey an inverse square law, by which the magnitude of the force is inversely proportional to the square of distance from its source. For example, gravity due to a point mass and electric force due to a point charge both follow such a law. You can describe inverse square forces as vector fields of the form

$$\vec{F}(x,y,z) = \langle \frac{-cx}{x^2+y^2+z^2}, \frac{-cy}{x^2+y^2+z^2}, \frac{-cz}{x^2+y^2+z^2} \rangle$$

where $\vec{F}$ is the force at point (x,y,z), c is a constant of proportionality, and force is measured in a coordinate system centered at the point source of the force.

Part 1. Show that force fields that obey the inverse square law are conservative.

Part 2. Consider the specific force field

$$\vec{F}(x,y,z) = \langle \frac{-x}{x^2+y^2+z^2}, \frac{-y}{x^2+y^2+z^2}, \frac{-z}{x^2+y^2+z^2} \rangle$$

Use the fundamental theorem of line integrals to integrate this field along the path defined parametrically by

$$\vec{r}(t) = \langle t, t, t^2 - 1 \rangle$$

as t ranges from 0 to 2.

Problem 3. (A variation on exercise 66 in section 6.2 of OpenStax *Calculus Volume 3*.)

Find the work done by force

$$\vec{F}(x,y) = \langle 2x, y \rangle$$

on a mass moving counterclockwise around the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$. Solve the problem in at least two and possibly three ways, namely:

1. Evaluate the relevant line integral directly
2. Use Green's theorem to evaluate the line integral
3. If $\vec{F}$ is conservative, use properties of conservative vector fields to find the integral without evaluating it explicitly. If this case applies, show that $\vec{F}$ is indeed conservative.

Problem 4. (Exercise 76 in section 6.2 of OpenStax *Calculus Volume 3*.)

Find

$$\int_C y^2\,dx + (xy - x^2)\,dy$$

where C is the curve $y = 3x$ between points $(0,0)$ and $(1,3)$.

Problem 5. (Exercise 166 in section 6.4 of OpenStax *Calculus Volume 3*.)

Use Green's theorem to prove that the area of a disk of radius a is $A = \pi a^2$.

Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the "Grade By" date above.

I will use the following guidelines in grading this problem set:

- What I expect (8 points). Your written solutions and verbal explanations show that you understand (1) how to evaluate line integrals of scalar functions, (2) how to evaluate line integrals of vector fields, (3) how to recognize conservative vector fields, (4) how to find potential functions for conservative vector fields, (5) how to use the fundamental theorem of line integrals to evaluate line integrals of vector fields, (6) how to use Green's theorem to evaluate line integrals of vector fields, and (7) how to apply line integrals in problem solving.

- Three quarters of what I expect (6 points). Plausible, but not exclusive, examples include failing to understand 1 or 2 of the expected items, OR having errors beyond arithmetic or typographical mistakes in one or more solutions even though you generally understand the expected items.

- Half of what I expect (4 points). Plausible but non-exclusive examples include showing that you understand 2 to 5 of the expected items, with no understanding of the others, OR showing that you partially but not completely understand all the expected items.

- Exceeding what I expect (typically 1 point added to what you otherwise earn). Generally, demonstrating that you have nontrivially engaged with math in ways beyond what is needed to solve the given problems exceeds my expectations. Exploring ways of using muPad or similar technology to calculate line integrals would be one way to do this.