Math 223 — Final Exam

May 9, 2016

General Directions. This is an open-book, open-notes, open-computer test. However, you may not communicate with any person, except me, during the test. You may use calculators and computer algebra systems on this test, but should indicate any parts of answers computed by them. You have three hours and 20 minutes in which to do the test. Put your answer to each question in the space provided (use the backs of pages if you need more space). Be sure to **show your work**! I give partial credit for incorrect answers if you show correct steps leading up to them; conversely, I do not give full credit even for correct answers if it is not clear that you understand where those answers come from. Good luck.

This test contains 8 questions, one with 2 parts, on 7 pages.

Question 1 (10 Points). A hypothetical computer animation system is generating a video of a soaring hawk. The hawk's position at time t is given by

$$\vec{r}(t) = \langle t + \cos t\, , 2 \sin t\, , 100 - t \rangle$$

as t ranges from 0 to 2π. In order to draw the hawk facing in the right direction, the animation system needs to know not only what the hawk's position is, but also what direction it is moving in. Derive an expression for the hawk's direction of motion at time t for $0 \le t \le 2\pi$.

<u>Solution</u>: Any vector tangent to **r(**t), i.e., parallel to the derivative of **r**, points in the hawk's direction of motion. The simplest such vector to find is **r**´, i.e.,

$$\overrightarrow{r'}(t) = \langle 1 - \sin t, 2 \cos t\, , -1 \rangle$$

Question 2. A bicyclist is riding around a circular track in such a manner that her position at time t is given by

$$\vec{r}(t) = \langle 100 \cos\left(\frac{t}{2}\right), 100 \sin\left(\frac{t}{2}\right)\rangle$$

as t varies from 0 to 4π. When she starts her ride, the wind is blowing so as to push her with a force of 10 pounds in the positive x direction, and keeps blowing like that until $t = 2\pi$. In other words, the bicyclist is riding into a headwind for the first half of her ride. Mathematically, the force from the wind can be described by the vector

$$\vec{F}_1 = \langle 10,0\rangle$$

Exactly when $t = 2\pi$, and the bicyclist finally expects to have a tailwind, the wind instantaneously shifts to push the bicyclist towards the positive y direction:

$$\vec{F}_2 = \langle 0,10\rangle$$

The rider finds herself wondering how much work she does against the wind over her entire (i.e., $0 \le t \le 4\pi$) ride.

<u>Part</u> <u>A</u> (10 Points). Can the rider use Green's Theorem to calculate the work? Explain why or why not in a sentence or two.

<u>Solution:</u> The rider cannot use Green's Theorem because her path through each vector field isn't closed. Or for a more technical answer, if you think of $\mathbf{F}_1$ and $\mathbf{F}_2$ as describing a single vector field that changes abruptly at $\mathbf{r}(2\pi)$, that vector field doesn't have continuous partial derivatives over the rider's path.

<u>Part</u> <u>B</u> (15 Points). Using any applicable method, calculate the total work the rider does against the wind.

<u>Solution</u>: We can find the work the rider does by integrating the force she exerts against the wind over the path she rides. Notice that the force the rider exerts against the wind is the negative of the force the wind exerts on her, and that there are different force vectors for different parts of her path. Thus the work she does is

$$W = \int_0^{2\pi} -\overrightarrow{F_1}(\vec{r}(t)) \cdot \overrightarrow{r'}(t)\, dt + \int_{2\pi}^{4\pi} -\overrightarrow{F_2}(\vec{r}(t)) \cdot \overrightarrow{r'}(t)\, dt$$

For both integrals,

$$\overrightarrow{r'}(t) = \langle -50 \sin\frac{t}{2}, 50 \cos\frac{t}{2}\rangle$$

so the work is

$$W = \int_0^{2\pi} \langle -10,0\rangle \cdot \langle -50\sin\frac{t}{2}, 50\cos\frac{t}{2}\rangle\, dt + \int_{2\pi}^{4\pi} \langle 0,-10\rangle \cdot \langle -50\sin\frac{t}{2}, 50\cos\frac{t}{2}\rangle\, dt$$

$$= \int_0^{2\pi} 500\sin\frac{t}{2}\, dt + \int_{2\pi}^{4\pi} -500\cos\frac{t}{2}\, dt$$

Using the substitution $u = t/2$ in both integrals gives

$$W = [-1000\cos u]_0^{\pi} - [1000\sin u]_{\pi}^{2\pi}$$

$$= 1000 - -1000$$

$$= 2000$$

So, assuming distances are measured in feet (and remembering that the question says force is measured in pounds), the rider does 2000 foot-pounds of work during her ride.

Question 3 (15 Points). Calculate the flux of $\vec{F}(x,y) = \langle xy^2, y\rangle$ across the boundary of the triangle enclosed in the lines $y = x$, $y = -x$, and $x = 1$.

Solution: Use Green's theorem to find the flux as an integral over the triangle. Bounds for the triangle range from 0 to 1 in the x dimension, and from $-x$ to x in the y dimension. So from the flux form of Green's theorem, and using our book's terminology for the components of **F**…

$$\Phi = \int_0^1 \int_{-x}^{x} P_x + Q_y \, dy \, dx$$

$$= \int_0^1 \int_{-x}^{x} y^2 + 1 \, dy \, dx$$

$$= \int_0^1 \left[\frac{y^3}{3} + y\right]_{-x}^{x} dx$$

$$= \int_0^1 \frac{2}{3}x^3 + 2x \, dx$$

$$= \left[\frac{1}{6}x^4 + x^2\right]_0^1$$

$$\frac{1}{6} + 1$$

$$= \frac{7}{6}$$

Question 4 (10 Points). A rectangle has one corner at point (1,2,3) and sides parallel to the vectors $\langle 2,1,2\rangle$ and $\langle -1,2,0\rangle$. Furthermore, the length of each side is the magnitude of the corresponding vector. Calculate the coordinates of the center of the rectangle.

Solution: Think of the vectors as giving offsets from (1,2,3) to the corners of the rectangle. Adding half of each vector to (1,2,3) will therefore give the coordinates of the center, i.e.,

$$(1,2,3) + \frac{1}{2}\langle 2,1,2\rangle + \frac{1}{2}\langle -1,2,0\rangle$$

$$= (1,2,3) + \langle 1, \tfrac{1}{2}, 1\rangle + \langle \tfrac{-1}{2}, 1,0\rangle$$

$$= \left(\tfrac{3}{2}, \tfrac{7}{2}, 4\right)$$

Question 5 (15 Points). Phineas Phoole and his brother Phileas are trying to calculate

$$\int_0^{\pi/2} \int_0^1 x \cos(xy)\, dx\, dy$$

Phineas wails and gnashes his teeth, saying that they will have to start by integrating $x \cos(xy)$, which will be hard to do. Phileas says no, the integral is equivalent to

$$\int_0^1 \sin\left(\frac{\pi}{2} x\right) dx$$

which won't be hard to evaluate at all.

Explain which brother is right, and why, in a sentence or two.

Solution: By Fubini's theorem, switching the order of integration doesn't change the value of the integral, so

$$\int_0^{\pi/2} \int_0^1 x \cos(xy)\, dx\, dy = \int_0^1 \int_0^{\pi/2} x \cos(xy) dy\, dx$$

Evaluate the inner integral by using the substitution $u = xy$, to get

$$\int_0^1 \int_0^{x\pi/2} \cos u\, du\, dx$$

$$= \int_0^1 \sin\left(\frac{\pi}{2} x\right) dx$$

So Phileas is right.

Question 6 (15 Points) Find the point (or points) on the curve $y = 2/x$ that is (or are) the closest to the origin of all the points on that curve.

Solution: We want to minimize distance (as given by the distance formula), subject to the constraint that $y = 2/x$, or, equivalently, $xy = 2$. Also note that minimizing the square of distance will also minimize distance, and provides simpler equations to work with. So we therefore set out to minimize $f(x,y) = x^2 + y^2$ subject to the constraint that $g(x,y) = xy - 2 = 0$. We can do this with Lagrange multipliers, as follows:

$$\nabla f = \langle 2x, 2y \rangle = \lambda \nabla g = \lambda \langle y, x \rangle$$

so in particular

$$2x = \lambda y$$
$$2y = \lambda x$$

Rewriting the first of these equations as

$$x = \frac{\lambda y}{2}$$

and substituting into the second we get

$$2y = \frac{\lambda^2 y}{2}$$

$$4y = \lambda^2 y$$

and so $\lambda = \pm 2$ or $y = 0$. Note that the constraint isn't satisfiable with $y = 0$, so we must have $\lambda = \pm 2$. Substituting this into the earlier equations we conclude that $x = \pm y$. Now substituting into the constraint equation $xy = 2$ yields $\pm x^2 = 2$; since $-x^2$ cannot be 2, we conclude that in fact $x = y = \pm\sqrt{2}$. The points on the curve $y = 2/x$ that are closest to the origin are therefore $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$.

Question 7 (15 Points). A certain sculpture consists of the metal film hanging perpendicular to the ground. In a coordinate system whose xy plane is the ground, whose positive z axis measures height above ground, and whose origin is at one corner of the sculpture, the sculpture's bottom edge rests on the curve $y = x^2$, for $0 \le x \le 1$, and its top edge rises with increasing x, specifically having height $z = x$. Calculate the area of one side of the sculpture. (If you want to attach units to your answer, assume distances are measured in feet.)

Solution: The line integral of z along the curve $y = x^2$ for $0 \le x \le 1$ will give the area. We can parameterize the curve as

$$\vec{r}(t) = \langle t, t^2 \rangle, 0 \le t \le 1$$

and so calculate area as

$$A = \int_0^1 z(\vec{r}(t)) \left|\vec{r'}(t)\right| dt$$

$$= \int_0^1 t |\langle 1,2t \rangle| \, dt$$

$$= \int_0^1 t\sqrt{1 + 4t^2} \, dt$$

Using the substitution $u = 1 + 4t^2$ and so $du = 8t\,dt$ transforms this integral into

$$A = \frac{1}{8} \int_1^5 \sqrt{u} \, du$$

$$= \frac{1}{8} \left[\frac{2}{3} u^{3/2} \right]_1^5$$

$$= \frac{1}{12} \left(5\sqrt{5} - 1 \right)$$

$$= \frac{5\sqrt{5} - 1}{12}$$

Question 8 (15 Points). In a science fiction movie so unmemorable that it might even be made up, Spaceman Spiff flies his star cruiser through the dangerous Farzman Field. The strength of the Farzman Field at point (x,y,z) (in some grand galactic coordinate system) is given by the function

$$F(x, y, z) = xy^2e^{-z}$$

Spiff's grand galactic coordinates are (2,3,0), and he is flying in direction $\langle 1,1,1 \rangle$. Calculate the instantaneous rate of change in the Farzman Field that Spiff experiences flying in this direction at this point. (If you want to attach units to your answer, the strength of the Field is measured in units called "Farzmans," and distances are measured in parsecs.)

Solution: Spiff needs the directional derivative of F in the direction of $\langle 1,1,1 \rangle$. This directional derivative can be calculated as $\nabla F \bullet \underline{\mathbf{u}}$ where **u** is a unit vector parallel to $\langle 1,1,1 \rangle$. So…

$$\nabla F \cdot \vec{u} = \langle y^2e^{-z}, 2xye^{-z}, -xy^2e^{-z} \rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$$

Evaluating this expression at point (2,3,0) yields

$$\langle 9,12,-18 \rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$$

$$= \frac{9 + 12 - 18}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}}$$

$$= \sqrt{3}$$