Math 223 — Hour Exam 2

March 31, 2016

General Directions. This is an open-book, open-notes, open-computer test. However, you may not communicate with any person, except me, during the test. You may use calculators and computer algebra systems on this test, but should indicate any parts of answers computed by them. You have the full class period (75 minutes) in which to do the test. Put your answer to each question in the space provided (use the backs of pages if you need more space). Be sure to **show your work**! I give partial credit for incorrect answers if you show correct steps leading up to them; conversely, I do not give full credit even for correct answers if it is not clear that you understand where those answers come from. Good luck.

This test contains 6 questions on 6 pages.

Question 1 (10 Points). Consider the hyperboloid

$$\frac{x^2}{4} - \frac{y^2}{9} - z^2 = 1$$

Show quantitatively (i.e., by finding and interpreting appropriate numbers, equations, vectors, etc.) that the tangent planes to this hyperboloid that pass through (2,0,0) and (-2,0,0) are parallel to each other.

Planes are parallel if their normals are, and the normal to an implicit surface is its gradient. So let $f(x,y,z) = \frac{x^2}{4} - \frac{y^2}{9} - z^2$ and show that $\nabla f|_{(2,0,0)} = k \nabla f|_{(-2,0,0)}$ for some constant k.

$$\nabla f = \left\langle \frac{x}{2}, \frac{-2y}{9}, -2z \right\rangle$$

$$\left.\begin{aligned} \nabla f|_{(2,0,0)} &= \langle 1, 0, 0 \rangle \\ \nabla f|_{(-2,0,0)} &= \langle -1, 0, 0 \rangle \end{aligned}\right\} \quad k = -1$$

Question 2 (10 Points). Let z be defined by the equation

$$z = xy^2 + 2x - y$$

Does z change more in response to a small change in the value of x, or to a small change in the value of y, around $x = 1$ and $y = 3$? Justify your choice in a sentence or two.

The changes in z as x and y change are given by $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, respectively. So find those derivatives at $(1,3)$ and see which is larger:

$$\left.\frac{\partial z}{\partial x}\right|_{(1,3)} = \left. y^2+2 \right|_{(1,3)} = 11$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,3)} = \left. 2xy - 1 \right|_{(1,3)} = 5$$

So z changes more in response to a change in x than to a change in y.

Question 3 (10 Points). A function $f(x,y)$ has level curves that are all straight lines of the form $y = x + c$, with different c values corresponding to different level curves. Suppose $f(0,2) = 1$. Find one other point (x,y) at which $f(x,y) = 1$. Explain your reasoning in a sentence or two.

$f(x,y)$ has a constant value for all points on the same level curve. So $f(x,y)=1$ at any other point on the same level curve as $(0,2)$. That level curve has equation $y = x+2$, so other points where $f(x,y)=1$ include $(1,3)$, $(-1,1)$, etc.

Question 4 (15 Points). Does the function

$$f(x,y) = 1 - (x-1)^2 + y^2$$

have any local maxima or minima? Explain your reasoning.

Generally, the first and second derivatives of f are helpful for finding extreme values:

$$\frac{\partial f}{\partial x} = -2(x-1) = -2x+2$$

$$\frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial^2 f}{\partial x^2} = -2$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

Because the 2nd derivatives are all constants, the discriminant test result $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -4 < 0$ tells you that any critical points this function has must be saddle points, not maxima or minima.

(You could stop with the above, but if you want to know if there are any critical points, solve

$$\frac{\partial f}{\partial x} = 0 \rightarrow -2(x-1) = 0 \rightarrow x-1=0 \rightarrow x=1$$

and

$$\frac{\partial f}{\partial y} = 0 \rightarrow 2y = 0 \rightarrow y=0$$

so there is a critical point at $(1,0)$)

Question 5 (15 Points). Consider

$$\lim_{(x,y)\to(1,1)} \frac{x(y-1)}{x^2-y^2}$$

Either find this limit, or show that it doesn't exist.

This limit doesn't exist. To see this, consider approaching $(1,1)$ along the lines $x=1$ and $y=1$.

Along $x=1$, the limit becomes $\lim_{y\to1} \frac{(y-1)}{1-y^2} =$

$$\lim_{y\to1} \frac{y-1}{(1-y)(1+y)} = \lim_{y\to1} \frac{-(1-y)}{(1-y)(1+y)} = \lim_{y\to1} \frac{-1}{1+y} = \frac{-1}{2}$$

Along $y=1$, the limit becomes $\lim_{x\to1} \frac{x(1-1)}{x^2-1^2} = \lim_{x\to1} \frac{0}{x^2-1} = 0$

Since $\frac{x(y-1)}{x^2-y^2}$ has different limits along different lines of approach to $(1,1)$, the limit doesn't exist.

Question 6 (15 Points). Suppose $f(x,y,z)$ is a function whose derivatives are

$$\frac{\partial f}{\partial x} = x + y - z \quad \frac{\partial f}{\partial y} = xy \quad \frac{\partial f}{\partial z} = \frac{z}{y}$$

Further suppose function $g(t)$ is defined as

$$g(t) = f(t, t^2, t^3)$$

Find dg/dt.

By the chain rule

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= (x+y-z)(1) + (xy)(2t) + \left(\frac{z}{y}\right)(3t^2)$$

$$= t + t^2 - t^3 + (t^3)(2t) + \left(\frac{t^3}{t^2}\right)(3t^2)$$

$$= t + t^2 - t^3 + 2t^4 + 3t^3$$

$$= 2t^4 + 2t^3 + t^2 + t$$