Problem Set 12 — Vector Line Integrals

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Math 223 01

Complete By Sunday, December 2
Grade By Tuesday, December 4

Purpose

This problem set mainly reinforces your understanding of line integrals of vector fields and their applications. It also begins to develop your understanding of conservative vector fields. Specific things I expect you to be able to do by the time you finish this problem set include ...

- Recognize alternative forms of vector line integral
- Evaluate line integrals of vector fields
- Use line integrals to solve work problems
- Use line integrals to calculate circulation
- Use line integrals to calculate flux
- Reason about conservative vector fields.

Background

Line integrals of vector fields are discussed in the second subsection ("Vector Line Integrals") of section 6.2 in our textbook. Applications to work are discussed in that subsection, and briefly at the end of the third subsection ("Applications of Line Integrals"). Flux and circulation are the final subsection ("Flux and Circulation") of section 6.2. We talked about these subjects in class on November 19 and 26.

The end of section 6.1 of the textbook introduces conservative vector fields as gradient vector fields. I expect to discuss this material in class on November 29.

Activity

Solve the following problems. I have to admit that an ulterior motive in picking some of these problems was to give you practice evaluating some harder integrals than we've seen so far in this course, so be particularly careful not to use computers, calculators, or tables of integrals in this problem set.

Problem 1. (An extension of exercise 76 in section 6.2 of OpenStax *Calculus Volume 3*.)

Part 1. Find

$$\int_C y^2\,dx + (xy - x^2)\,dy$$

where C is the curve $y = 3x$ between points $(0,0)$ and $(1,3)$.

Part 2. Express the integral in Part 1 in the form

$$\int_C \vec{F} \cdot d\vec{r}$$

showing the vector field $\vec{F}$ explicitly.

Problem 2. (Inspired by exercise 68 in section 6.2 of OpenStax *Calculus Volume 3*.)

Find the work done by the force field

$$\vec{F}(x,y) = \langle xe^x, -ye^x \rangle$$

on a particle that moves along the path $\vec{r}(t) = \langle \sin t, \cos t \rangle$ as t ranges from 0 to $\frac{\pi}{2}$.

Problem 3. Let $\vec{F} = \langle -y^2, x^2 \rangle$, and let C be the curve parameterized by $\vec{r}(t) = \langle \cos t, \sin t \rangle, 0 \leq t \leq 2\pi$.

Part 1. Find the circulation of $\vec{F}$ around C.

Part 2. Find the flux of $\vec{F}$ across C.

Part 3. Find the flux of $\vec{F}$ across the top half of C (i.e., the curve parameterized by $\vec{r}(t) = \langle \cos t, \sin t \rangle, 0 \leq t \leq \pi$).

Problem 4. Suppose $\vec{F}$ and $\vec{G}$ are any two conservative vector fields. Show that ...

Part 1. Vector field $\vec{F} + \vec{G}$ is conservative.

Part 2. Vector field $k\vec{F}$, where k is any scalar, is conservative.

Hint: Use the fact that conservative vector fields are gradient fields.

Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the "Grade By" date above. If you work in a group on this problem set, all members of the group can come to the same meeting.

I will use the guidelines discussed in class on November 12 to grade this problem set.