Math 223 — Hour Exam 2

November 2, 2015

General Directions. This is an open-book, open-notes, open-computer test. However, you may not communicate with any person, except me, during the test, and you may not use calculators or computer systems that perform symbolic math (e.g., systems that can calculate derivatives or integrals, solve algebraic equations, etc). You have the full class period (75 minutes) in which to do the test. Put your answer to each question in the space provided (use the backs of pages if you need more space). Be sure to **show your work**! I give partial credit for incorrect answers if you show correct steps leading up to them; conversely, I do not give full credit even for correct answers if it is not clear that you understand where those answers come from. Good luck.

This test contains 5 questions (2 with two parts) on 5 pages.

Question 1 (10 Points). Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, given that

$$f(x, y) = \sin\left(\frac{x^2}{y}\right)$$

Both derivatives use the chain rule, multiplying the derivative of sine (i.e., cosine) by the derivative with respect to *x* or *y* of the argument to sine. Differentiating with respect to y is easiest if you think of x^2/y as a constant (for purposes of this derivative) times y^{-1}.

$$\frac{\partial f}{\partial x} = \frac{2x}{y} \cos\left(\frac{x^2}{y}\right)$$

$$\frac{\partial f}{\partial y} = \left(\frac{-x^2}{y^2}\right) \cos\left(\frac{x^2}{y}\right)$$

Question 2. In a strange alternate universe, the scenic village of Geneseo, New York is widely known as the home of the Geneseo Widget Works, makers of fine hand-crafted solid gold widgets. In this universe, the number of widgets the Widget Works can make per day is given by the function

$$W(a, g) = 3a + \frac{g}{2} - \frac{a^2}{100}$$

where a is the number of widget-making artisans at the Widget Works, g is the amount of gold available, and $W(a,g)$ is the number of widgets made.

At a certain point in time, the Widget Works employs 100 artisans and has 500 units of gold on hand, so they produce 450 widgets per day.

Part A (10 Points). At this point in time, the manager of the Widget Works decides to hire more artisans and buy more gold. In particular, she decides to hire artisans and increase her gold supply at equal rates (i.e., for every artisan she hires, she will increase her gold supply by one unit of gold). What is the instantaneous rate of change in the number of widgets produced per day for this expansion strategy at this point in time?

The directional derivative in the direction of the change gives the instantaneous rate. The direction vector, **u**, is a unit vector in the direction <1,1>, i.e., <√2/2,√2/2>. The directional derivative is then

$$\left.\frac{dW}{ds}\right|_{\boldsymbol{u},(100,500)} = \nabla W|_{(100,500)} \cdot \boldsymbol{u} = \left.\langle 3 - \frac{a}{50}, \frac{1}{2}\rangle\right|_{(100,500)} \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$$

$$= \langle 1, \frac{1}{2}\rangle \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle = \frac{3\sqrt{2}}{4}$$

Part B (10 Points). Find a unit-length vector in the artisans-and-gold space that indicates the “direction” in which the manager should change the number of artisans and/or gold supply in order to get the maximum increase in production.

The direction of greatest change is the direction the gradient points in. From Part A, the gradient at (100,500) is <1, ½>. This vector’s magnitude is

$$\sqrt{1 + \left({}^{1}\!/_{2}\right)^2} = \sqrt{{}^{5}\!/_{4}} = \frac{\sqrt{5}}{2}$$

The unit vector is thus

$$\frac{2}{\sqrt{5}}\langle 1, \frac{1}{2}\rangle = \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle = \langle \frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\rangle$$

Question 3 (10 Points). A certain function, $f(x,y)$, has a level curve defined by the equation $y = x - 2$. Furthermore, $f(1,-1) = 10$. What, if anything, do you know about the values of $f(1,0)$ and $f(2,0)$? Explain your reasoning in a sentence or two.

(1,-1) is on the level curve since 1-2 = -1; so is (2,0), but (1,0) is not. Thus you know that $f(2,0) = 10$, but know nothing about $f(1,0)$.

Question 4. Consider the function

$$f(x, y) = \frac{x - 3y}{xy}$$

<u>Part A</u> (10 Points). What is the domain of this function? Explain your answer in a sentence or two.

This function is defined everywhere the denominator is non-zero, i.e., everywhere except along the lines $x = 0$ and $y = 0$. The domain is thus the entire *xy* plane except for these lines.

<u>Part B</u> (10 Points). Find $\lim_{(x,y)\to(0,0)} f(x, y)$, or show that the limit does not exist.

The limit doesn't exist. Consider approaching (0,0) along the line $y = x$ from positive x values. Along this line, $f(x,y)$ is equivalent to

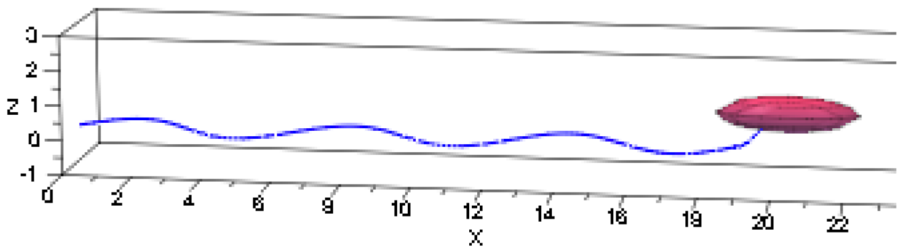
$$\frac{x - 3x}{x^2} = \frac{-2x}{x^2} = \frac{-2}{x}$$

and so the limit is $-\infty$. On the other hand, along the line $y = x/3$ the function is equivalent to

$$\frac{x - 3\left(x/3\right)}{x^2/3} = \frac{0}{x^2}$$

and so the limit is 0.

Question 5 (15 Points). I was recently playing with our computer graphic snake, and decided to give it a head. The head is an ellipsoid, while the body and neck are still curves, like so:



Ignoring some constants that position the head relative to the rest of the snake, the basic equation that defines the head is

$$\left(\frac{x}{2}\right)^2 + (2y)^2 + (2z)^2 = 1$$

Define the snake's nose to be the point ($\sqrt{3}, 0, \frac{1}{4}$). In what direction does the normal to the nose point?*

Identify the left side of the head equation as $f(x,y,z)$. The tangent plane to the nose then has equation

$$\frac{\partial f}{\partial x}\left(x - \sqrt{3}\right) + \frac{\partial f}{\partial y} y + \frac{\partial f}{\partial z}\left(z - \frac{1}{4}\right) = 0$$

which has normal vector

$$\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle \frac{x}{2}, 8y, 8z \rangle$$

Evaluating at ($\sqrt{3}, 0, \frac{1}{4}$) yields < $\sqrt{3}/2$, 0, 2 > as the direction of the normal. (A unit direction vector would be < $\sqrt{3}/\sqrt{19}$, 0, $4/\sqrt{19}$ > = < $\sqrt{57}/19$, 0, $4\sqrt{19}/19$ >)

* Someone writing a computer program to draw the snake would actually need to know this in order to get lighting effects right on the head; muPad probably computes normals in order to shade the surfaces it plots.