Math 223 — Hour Exam 1 Solution

October 5, 2015

General Directions. This is an open-book, open-notes, open-computer test. However, you may not communicate with any person, except me, during the test, and you may not use calculators or computer systems that perform symbolic math (e.g., systems that can calculate derivatives or integrals, solve algebraic equations, etc). You have the full class period (75 minutes) in which to do the test. Put your answer to each question in the space provided (use the backs of pages if you need more space). Be sure to **show your work**! I give partial credit for incorrect answers if you show correct steps leading up to them; conversely, I do not give full credit even for correct answers if it is not clear that you understand where those answers come from. Good luck.

This test contains 5 questions (2 with 2 parts) on 5 pages.

Question 1 (5 Points). Give a point that lies on the paraboloid $y^2 + 4z^2 = x$. Explain the reasoning behind your choice in a sentence or two.

Pick values for y and z, and use the equation of the paraboloid to find x, e.g., let $y = z = 1$, then $y^2 + 4z^2 = 5 = x$. Thus the point (5,1,1) is on the paraboloid.

Question 2. An airplane flying a loop-the-loop follows a path defined by the equation

$$\mathbf{r}(t) = < 15\cos(2t), 10t, 15\sin(2t) >, 0 \le t \le \pi$$

Assume that t measures time in seconds and that distances are measured in meters.

Part A (10 Points). In what direction is the airplane moving when $t = \pi/4$? How fast is it moving then?

The derivative of **r** gives the direction, and its magnitude the speed:

$$\frac{d\boldsymbol{r}}{dt} = < -30\sin 2t, 10, 30\cos 2t >$$

$$\left.\frac{d\boldsymbol{r}}{dt}\right]_{t=\frac{\pi}{4}} = < -30\sin\frac{\pi}{2}, 10, 30\cos\frac{\pi}{2} > = < -30, 10, 0 >$$

The magnitude of this vector is

$$\sqrt{30^2 + 10^2} = \sqrt{1000} = 10\sqrt{10}$$

Part B (15 Points). How long (in meters) is this part of the airplane's flight?

The general formula for the magnitude of the plane's velocity is

$$|\boldsymbol{v}| = \sqrt{(-30\sin 2t)^2 + 10^2 + (30\cos 2t)^2} = \sqrt{900\sin^2(2t) + 100 + 900\cos^2(2t)} = \sqrt{900 + 100} = \sqrt{1000} = 10\sqrt{10}$$

The arc length of the curve is then

$$s = \int_0^{\pi} 10\sqrt{10}\, dt = \left[10\sqrt{10}t\right]_0^{\pi} = 10\pi\sqrt{10}$$

meters.

Question 3. Sometimes you find yourself in a situation where you have a vector **v**, and want to find another vector that is perpendicular to **v**. (For example, when I suggested how one might draw a computer graphic snake, I talked about using two vectors perpendicular to the tangent to the snake to put a "body" around it; the first of these vectors can really be any perpendicular to the tangent, it doesn't have to be the specific one I calculated.) Phineas Phoole suggests that if $\mathbf{v} = < a, b, c >$, then the vector $\mathbf{u} = < b\text{-}c, c\text{-}a, a\text{-}b >$ is orthogonal to **v** (which, he further claims, is "pretty much the same as being perpendicular")

<u>Part</u> <u>A</u> (10 Points). Is Phineas right that his **u** vector is orthogonal to **v**? Be sure to explain the reasoning behind your answer, don't just say "yes" or "no."

Vectors are orthogonal if their dot product is 0: $\mathbf{v}\cdot\mathbf{u} = <a,b,c>\cdot<b\text{-}c,c\text{-}a,a\text{-}b> = a(b\text{-}c) + b(c\text{-}a) + c(a\text{-}b) = ab - ac + bc - ab + ac - bc = 0$. So yes, Phineas's construction does create a vector orthogonal to **v**.

<u>Part</u> <u>B</u> (10 Points). Most schemes for constructing a vector perpendicular to **v** will, for some values of **v**, construct the 0 vector, $<0,0,0>$, which is orthogonal to **v** but not perpendicular to it (i.e., you can't really talk about the angle between the 0 vector and **v** at all, let alone whether it is 90 degrees—Phineas is wrong if he thinks that "orthogonal" and "perpendicular" are synonyms). For what, if any, vectors **v** does Phineas's scheme produce the 0 vector?

$< b\text{-}c, c\text{-}a, a\text{-}b > = <0,0,0>$ if $a = b = c$, so Phineas will generate the 0 vector for any **v** parallel to the diagonal line $x = y = z$ (and for the 0 vector itself).

Question 4 (15 Points). The curve

$$\boldsymbol{r}(t) = < t, \frac{1}{2}t^2, \frac{\sqrt{3}}{2}t^2 >$$

lies in a single plane. Give an equation for this plane.

The strategy is to find 3 points on the curve; the cross product of the vectors between 2 pairs of them is then the normal to the curve, which can be used with one of the points to finish the equation for the plane.

To find points on the curve, plug any 3 convenient t values into $\mathbf{r}(t)$, say $t = -1$, $t = 0$, $t = 1$. $\mathbf{r}(-1) = < -1, ½, \sqrt{3}/2 >$; $\mathbf{r}(0) = < 0, 0, 0 >$; $\mathbf{r}(1) = < 1, ½, \sqrt{3}/2 >$.

The vectors from $\mathbf{r}(0)$ to $\mathbf{r}(-1)$ and $\mathbf{r}(0)$ to $\mathbf{r}(1)$ are $< -1, ½, \sqrt{3}/2 >$ and $< 1, ½, \sqrt{3}/2 >$ respectively. Their cross product is

$$< \left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right), -\left(\frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right), \left(\frac{-1}{2} - \frac{1}{2}\right) > = < 0, \sqrt{3}, -1 >$$

The plane's equation is thus $\sqrt{3}y - z = D$ for some as-yet unknown D. Plugging point (0,0,0) (which I found earlier to be on the curve and so in the plane) into this equation yields $D = \sqrt{3} \times 0 - 0 = 0$. So the final equation for the plane is

$$\sqrt{3}y - z = 0$$

Question 5 (10 Points). Find **r**(t), given that

$$\frac{d\boldsymbol{r}}{dt} = < -\sin t\,, 4t^3 + t, 2e^t >$$

and

$$\boldsymbol{r}(0) = < 1, 0, 2 >$$

Start by finding **r**(t) as

$$\boldsymbol{r}(t) = \int < -\sin t, 4t^3 + t, 2e^t > dt = < \cos t\,, t^4 + \frac{1}{2}t^2, 2e^t > + \boldsymbol{C}$$

Then

$$\boldsymbol{r}(0) = < 1,0,2 > = < \cos 0\,, 0, 2e^0 > + \boldsymbol{C} = < 1,0,2 > + \boldsymbol{C}$$

So **C** = <0,0,0>.