Math 221 03
Prof. Doug Baldwin

# Last(!) Problem Set — Integrals

Complete by **Monday, May 6**
Grade by **Friday, May 10**

## Purpose

This problem set develops your understanding of definite and indefinite integrals and methods for evaluating them. By the time you finish this problem set you should be able to ...

- Evaluate definite integrals using antiderivatives
- Evaluate definite and indefinite integrals using substitution
- Use Mathematica to evaluate definite and indefinite integrals
- Use definite integrals to find average values of functions and areas between graphs and the axis
- Use antidifferentiation rules to evaluate integrals.

## Background

This problem set is based on sections 5.2, 5.3, and 5.5 of our textbook. We discussed (or will discuss) this material in classes between April 24 and approximately May 1.

## Activity

Solve the following problems:

**Question 1.** Find the value of

$$\int_1^2 3x^2 + 1 - \frac{1}{x^2}\,dx$$

After calculating this integral by hand, use Mathematica or similar technology to check your answer.

**Solution:**

$$\begin{aligned}\int_1^2 3x^2 + 1 - \frac{1}{x^2}\,dx &= \left[x^3 + x + \frac{1}{x}\right]_1^2 \\ &= \left(2^3 + 2 + \frac{1}{2}\right) - \left(1^3 + 1 + \frac{1}{1}\right) \\ &= 10\frac{1}{2} - 3 \\ &= 7\frac{1}{2}\end{aligned}$$

**Question 2.** Consider the function $f(x) = x^2$ over the interval $-1 \le x \le 1$.

**Part A.** Use a definite integral to find the average value of $f(x)$ over the interval.

**Solution:**

$$\begin{aligned}
\bar{y} &= \frac{1}{1-(-1)} \int_{-1}^{1} x^2\,dx \\
&= \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} \\
&= \frac{1}{2}\left(\frac{1}{3} - \frac{-1}{3}\right) \\
&= \frac{1}{2} \times \frac{2}{3} \\
&= \frac{1}{3}
\end{aligned}$$

**Part B.** How tall does a rectangle extending from $x = -1$ to $x = 1$ have to be in order to have the same area as the area between the graph of $f(x)$ and the $x$ axis between $x = -1$ and $x = 1$? Be sure you can explain how you arrived at your answer, including any connections you see between the area of this rectangle, the area under the graph, and the average value of the function.

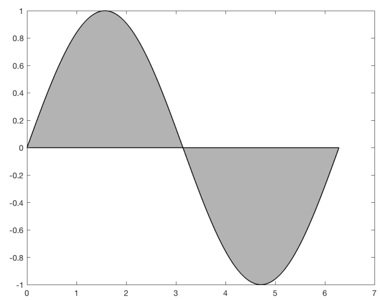
**Solution:** The area under the graph is

$$\begin{aligned}
A &= \int_{-1}^{1} x^2\,dx \\
&= \left[\frac{x^3}{3}\right]_{-1}^{1} \\
&= \left(\frac{1}{3} - \frac{-1}{3}\right) \\
&= \frac{2}{3}
\end{aligned}$$

Since the interval between $x = -1$ and $x = 1$ is 2 units long, a rectangle that long would have to have a height of $\frac{1}{3}$ to have the same area.

This height is the average value of $f(x)$ over the interval, and illustrates a geometric way of interpreting the idea of "average value" of a continuous function: the average is the value that a constant function (or constant-height rectangle) would need to have in order to have the same area below it as the function does.

**Question 3.** Find the area between the $x$ axis and one cycle of a sine wave, i.e., the shaded area in this plot:



Also use Mathematica or similar technology to check your answer.

**Solution:** The area is basically the integral of $\sin x$, except that since $\sin x$ is negative over part of the interval, the integral has to be broken into two to avoid positive and negative areas cancelling:

$$\begin{aligned} A &= \int_0^\pi \sin x\,dx - \int_\pi^{2\pi} \sin x\,dx \\ &= [-\cos x]_0^\pi - [-\cos x]_\pi^{2\pi} \\ &= (1-(-1)) - ((-1)-1) \\ &= 2-(-2) \\ &= 4 \end{aligned}$$

**Question 4.** Evaluate each of the following definite or indefinite integrals. After evaluating them by hand, use Mathematica or similar technology to check your answers.

**Part A.**

$$\int 3x^2(1+x^3)\,dx$$

**Solution:** Use the substitution $u = 1 + x^3, du = 3x^2 dx$:

$$\begin{aligned} \int 3x^2(1+x^3)\,dx &= \int u\,du \\ &= \frac{u^2}{2} + C \\ &= \frac{(1+x^3)^2}{2} + C \end{aligned}$$

**Part B.** OpenStax *Calculus, Volume 1*, Problem 292 in Section 5.5:

$$\int_0^1 x\sqrt{1-x^2}\,dx$$

**Solution:** Use the substitution $u = 1 - x^2, du = -2xdx$:

$$\begin{aligned}\int_0^1 x\sqrt{1-x^2}\,dx &= \frac{-1}{2}\int_1^0 \sqrt{u}\,du \\ &= \frac{-1}{2}\int_1^0 u^{\frac{1}{2}}\,du \\ &= \frac{-1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^0 \\ &= \frac{-1}{2}\left(0-\frac{2}{3}\right) \\ &= \frac{1}{3}\end{aligned}$$

**Part C.** OpenStax *Calculus, Volume 1*, Problem 280 in Section 5.5:

$$\int t^2\cos^2(t^3)\sin(t^3)\,dt$$

**Solution:** First, use the substitution $u = t^3, du = 3t^2dt$:

$$\int t^2\cos^2(t^3)\sin(t^3)\,dt = \frac{1}{3}\int \cos^2 u \sin u\,du$$

To continue, use the substitution $v = \cos u, dv = -\sin u\,du$:

$$\begin{aligned}\frac{1}{3}\int \cos^2 u\sin u\,du &= -\frac{1}{3}\int v^2\,dv \\ &= \frac{-1}{3}\left(\frac{v^3}{3}\right)+C \\ &= \frac{-v^3}{9}+C \\ &= \frac{-\cos^3 u}{9}+C \\ &= \frac{-\cos^3(t^3)}{9}+C\end{aligned}$$

**Question 5.** Prove the following extension to the power rule for antiderivatives: For all constants $n \neq -1$ and $k$,

$$\int (x+k)^n\,dx = \frac{1}{n+1}(x+k)^{n+1}+C$$

**Solution:** Use the substitution $u = x + k, du = dx$:

$$\begin{aligned}
\int (x+k)^n \, dx &= \int u^n \, du \\
&= \frac{1}{n+1} u^{n+1} + C \\
&= \frac{1}{n+1} (x+k)^{n+1} + C
\end{aligned}$$

## Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the "Grade By" date above.