Math 221 03
Prof. Doug Baldwin

Problem Set 8 — Sums

Complete by **Wednesday, April 24**
Grade by **Monday, April 29**

Purpose

This problem set develops your understanding of definite integrals as Riemann sums, and of the underlying summation rules. By the time you finish this problem set you should be able to ...

- Evaluate summations
- Simplify summations
- Use Mathematica to evaluate and simplify summations
- Evaluate definite integrals as Riemann sums
- Use integration rules to evaluate definite integrals.

Background

This problem set is based on sections 5.1 and 5.2 of our textbook. We discussed (or will discuss) this material in classes between April 15 and 19.

Activity

Solve the following problems:

Question 1. Find the values of the following sums:

Part A.

$$\sum_{i=1}^{4} i$$

Solution:

$$\begin{aligned}\sum_{i=1}^{4} i &= \frac{4(4+1)}{2} \\ &= \frac{4 \times 5}{2} \\ &= 10\end{aligned}$$

Alternatively, just add up the terms: $\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$.

Part B. (Problem 10 in section 5.1 of OpenStax *Calculus, Volume 1.*) You may use a calculator for the purely numeric parts of this problem, but not to simplify the sum.

$$\sum_{j=11}^{20}(j^2-10j)$$

Solution:

$$\begin{aligned}
\sum_{j=11}^{20}(j^2-10j) &= \sum_{j=11}^{20} j^2 - 10\sum_{j=11}^{20} j \\
&= \sum_{j=1}^{20} j^2 - \sum_{j=1}^{10} j^2 - 10\left(\sum_{j=1}^{20} j - \sum_{j=1}^{10} j\right) \\
&= \frac{20\times 21\times 41}{6} - \frac{10\times 11\times 21}{6} - 10\left(\frac{20\times 21}{2} - \frac{10\times 11}{2}\right) \\
&= 10\times 7\times 41 - 5\times 11\times 7 - 10(10\times 21 - 5\times 11) \\
&= 935
\end{aligned}$$

Part C. Check your answers to the first two parts by evaluating the same sums with Mathematica or similar technology.

Question 2. Simplify the following sums, i.e., find expressions that give their values in terms of n with no explicit summation.

Part A.

$$\sum_{i=1}^{n}(i+2)$$

Solution:

$$\begin{aligned}
\sum_{i=1}^{n}(i+2) &= \sum_{i=1}^{n} i + \sum_{i=1}^{n} 2 \\
&= \frac{n(n+1)}{2} + 2n \\
&= \frac{n^2+n+4n}{2} \\
&= \frac{n^2+5n}{2}
\end{aligned}$$

Part B.

$$\sum_{i=2}^{n}(6i^2+2i)$$

Solution:

$$\begin{aligned}
\sum_{i=2}^{n}(6i^2+2i) &= \sum_{i=1}^{n}(6i^2+2i)-(6\times 1^2+2\times 1)\\
&= \sum_{i=1}^{n}(6i^2+2i)-8\\
&= \sum_{i=1}^{n}6i^2+\sum_{i=1}^{n}2i-8\\
&= 6\sum_{i=1}^{n}i^2+2\sum_{i=1}^{n}i-8\\
&= 6\frac{n(n+1)(2n+1)}{6}+2\frac{n(n+1)}{2}-8\\
&= n(n+1)(2n+1)+n(n+1)-8\\
&= (n^2+n)(2n+1)+n^2+n-8\\
&= 2n^3+n^2+2n^2+n+n^2+n-8\\
&= 2n^3+4n^2+2n-8
\end{aligned}$$

Part C. Check your results from the first two parts by simplifying the sums in Mathematica or a similar technology.

Question 3. Part A. Find $\int_1^2 x\,dx$ by evaluating the limit of a Riemann sum. Check your answer against what you get using area formulas from geometry.

Solution:

$$\begin{aligned}
\int_1^2 x\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}x\Delta x\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\frac{1}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{n}+\frac{i}{n^2}\right)\\
&= \lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}(n+i)\\
&= \lim_{n\to\infty}\frac{1}{n^2}\left(\sum_{i=1}^{n}n+\sum_{i=1}^{n}i\right)\\
&= \lim_{n\to\infty}\frac{1}{n^2}\left(n^2+\frac{n^2+n}{2}\right)\\
&= \lim_{n\to\infty}\frac{1}{n^2}\left(\frac{3}{2}n^2+\frac{n}{2}\right)\\
&= \lim_{n\to\infty}\frac{3}{2}+\frac{1}{2n}\\
&= \frac{3}{2}
\end{aligned}$$

The region in question is a trapezoid with width 1 and heights 1 and 2, so its area should be $1 \times \frac{1+2}{2} = \frac{3}{2}$.

Part B. Find $\int_0^1 x^2\,dx$ by evaluating the limit of a Riemann sum.

Solution:

$$\begin{aligned}
\int_0^1 x^2\,dx &= \lim_{x\to\infty}\sum_{i=1}^{n} x^2\Delta x \\
&= \lim_{x\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\frac{1}{n} \\
&= \lim_{x\to\infty}\sum_{i=1}^{n}\frac{i^2}{n^3} \\
&= \lim_{x\to\infty}\frac{1}{n^3}\sum_{i=1}^{n} i^2 \\
&= \lim_{x\to\infty}\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \\
&= \lim_{x\to\infty}\frac{1}{n^3}\frac{2n^3+3n^2+n}{6} \\
&= \lim_{x\to\infty}\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2} \\
&= \frac{1}{3}
\end{aligned}$$

Part C. Using your answers to Parts A and B, and the textbook's discovery (in Example 5.7) that $\int_0^2 x^2 dx = \frac{8}{3}$, find

$$\int_1^2 3x^2 - 2x\,dx$$

without taking further limits of Riemann sums (and without using the Fundamental Theorem of Calculus, i.e., without using the idea that definite integrals can be evaluated via antiderivatives).

Solution:

$$\begin{aligned}
\int_1^2 3x^2 - 2x\,dx &= 3\int_1^2 x^2\,dx - 2\int_1^2 x\,dx \\
&= 3\left(\int_0^2 x^2\,dx - \int_0^1 x^2\,dx\right) - 2\int_1^2 x\,dx \\
&= 3\left(\frac{8}{3}-\frac{1}{3}\right) - 2\times\frac{3}{2} \\
&= (8-1)-3 \\
&= 4
\end{aligned}$$

Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the "Grade By" date above.