Math 221 03
Prof. Doug Baldwin

Problem Set 6 — Derivative Applications 1

Complete by **Sunday, April 7**
Grade by **Wednesday, April 10**

Purpose

This problem set develops your understanding of certain applications of derivatives, including linear approximation, extreme values, and curve sketching. By the time you finish this problem set you should be able to ...

- Use derivatives to make linear estimates of a function's value
- Use derivatives to find minimum and maximum values of a function
- Use derivatives to estimate the shape of a function's graph
- Use standard differentiation rules to find derivatives.

Background

Chapter 4 of our textbook discusses applications of derivatives; this problem set focuses on material from sections 4.2 through 4.5. We discussed, or will discuss, that material in classes between March 15 and approximately April 3.

Activity

Solve the following problems:

Question 1. (Based on OpenStax *Calculus, Volume 1*, section 4.2, exercise 59)

Use linear approximation to estimate the value of $15.99^{\frac{1}{4}}$ (15.99 raised to the $\frac{1}{4}$ power). Decide for yourself what function to do an approximation of, and near what value. You may use a calculator for the resulting numeric calculations. You should also use a calculator to see how close your linear estimate is to the machine-calculated value.

Solution: Use $f(x) = x^{\frac{1}{4}}$ as the function, evaluated at $x = 16$. Then $f(16) = 2$, and $f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$, so $f'(16) = \frac{1}{32}$. Plugging into the linear approximation formula gives

$$\begin{aligned} f(15.99) &\approx f(16) + f'(16)(15.99 - 16) \\ &= 2 + \frac{1}{32} \times (-0.01) \\ &\approx 1.9996875 \end{aligned}$$

A calculator gives $f(15.99) = 1.9996874$.

Question 2. (Based on OpenStax *Calculus, Volume 1*, section 4.3, exercise 90)

It came out in one of our class discussions that every 2nd degree polynomial, i.e., every function of the form $y = ax^2 + bx + c$, has exactly one maximum or minimum. You may also have heard that this maximum or minimum occurs at $x = \frac{-b}{2a}$. Use calculus to derive this result.

Solution: $\frac{dy}{dx} = 2ax + b$. The maximum or minimum must occur where this equals 0, i.e., where

$$\begin{aligned} 2ax + b &= 0 \\ 2ax &= -b \\ x &= \frac{-b}{2a} \end{aligned}$$

Question 3. (Based on OpenStax *Calculus, Volume 1*, section 4.3, exercises 142 and 143)

Assume that during the California gold rush, the amount of gold per year produced t years after the gold rush started was

$$G(t) = \frac{25t}{t^2 + 16}$$

million ounces. When did the local and absolute maximum and minimum gold production occur, and what were the production levels? Assume the gold rush lasted 40 years, so that $0 \leq t \leq 40$. You may use a calculator for numeric calculations. (See the book's discussion just before exercises 142 and 143 on page 378 of our PDF version for more information.)

Solution: Find local minima and maxima by finding critical points for $G(t)$.

$$\begin{aligned} G'(t) &= \frac{25(t^2+16) - (25t)(2t)}{(t^2+16)^2} \\ &= \frac{25t^2 + 400 - 50t^2}{(t^2+16)^2} \\ &= \frac{400 - 25t^2}{(t^2+16)^2} \end{aligned}$$

This derivative is defined everywhere on the interval $[0, 40]$, so we just need to find places where it is 0:

$$\begin{aligned} \frac{400 - 25t^2}{(t^2+16)^2} &= 0 \\ 400 - 25t^2 &= 0 \\ 25t^2 &= 400 \\ t^2 &= 16 \\ t &= \pm 4 \end{aligned}$$

Since $t = -4$ isn't in the interval of interest, we just need to consider $t = 4$ as a critical point.

Evaluating $G(t)$ at $t = 4$ and both endpoints of the interval yields

$$\begin{aligned} G(0) &= 0 \\ G(4) &= 3.125 \\ G(40) &\approx 0.619 \end{aligned}$$

So the absolute minimum production was 0 at the beginning of the gold rush, and the absolute maximum was 3.125 million ounces 4 years after the gold rush started; this was also the only local maximum.

Question 4. (OpenStax *Calculus, Volume 1*, section 4.5, exercise 216) Sketch a graph of a function $f(x)$ that has the following features over the interval $[-3, 3]$:

- $f(x) > 0$ everywhere in the interval
- $f'(x) > 0$ for $x < 0$
- $f'(x) = 0$ for $0 < x < 1$
- $f'(x) > 0$ for $x > 1$

Solution:

Question 5. (OpenStax *Calculus, Volume 1*, section 4.5, exercise 218) Sketch a graph of a function $f(x)$ that has the following features over the interval $[-3, 3]$:

- f has a local maximum at $x = 0$
- f has local minima at $x = -2$ and $x = 2$
- $f''(x) > 0$ for $x < -1$
- $f''(x) < 0$ for $-1 < x < 1$
- $f''(x) > 0$ for $x > 1$

Solution:

Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the "Grade By" date above.