Math 221 03
Prof. Doug Baldwin

Problem Set 4 — Derivative Topics

Complete by **Tuesday, March 5**
Grade by **Friday, March 8**

Purpose

This problem set develops ideas of derivatives beyond those in problem set 3. In particular, by the time you finish this problem set you should be able to . . .

- Find derivatives of trigonometric functions
- Find 2nd and higher order derivatives
- Find antiderivatives
- Use the common differentiation rules in finding derivatives
- Use the chain rule in finding derivatives.

Background

This problem set is based on material in sections 3.3 through 3.6 of our textbook. We discussed this material in class between February 15 and 27.

Activity

Solve the following problems:

Question 1. Find the following derivatives.

1. $\frac{dz}{dx}$ if $z = 2\sin x - 3\tan x$
2. $h'(a)$ if $h(a) = (a^2 + 5)\cos a$
3. $\frac{ds}{dt}$ if $s = (t + 1)\frac{t^2-1}{t^2+2}$

Solution:

$$\frac{dz}{dx} = 2\cos x - 3\sec^2 x \quad (1)$$

$$h'(a) = 2a\cos a - (a^2 + 5)\sin a \quad (2)$$

$$
\begin{aligned}
\frac{ds}{dt} &= \frac{d}{dt}(t+1)\,\frac{t^2-1}{t^2+2} + (t+1)\frac{d}{dt}\left(\frac{t^2-1}{t^2+2}\right) \\
&= \frac{t^2-1}{t^2+2} + (t+1)\frac{2t(t^2+2)-2t(t^2-1)}{(t^2+2)^2} \\
&= \frac{t^2-1}{t^2+2} + (t+1)\frac{2t^3+4t-2t^3+2t}{(t^2+2)^2} \\
&= \frac{t^2-1}{t^2+2} + (t+1)\frac{6t}{(t^2+2)^2} \\
&= \frac{(t^2-1)(t^2+2)+6t(t+1)}{(t^2+2)^2} \\
&= \frac{t^4+2t^2-t^2-2+6t^2+6t}{(t^2+2)^2} \\
&= \frac{t^4+7t^2+6t-2}{(t^2+2)^2}
\end{aligned} \tag{3}
$$

Question 2. Find the second derivatives of z and $h(a)$ from Question 1, i.e., find $\frac{d^2z}{dx^2}$ and $h''(a)$ where z and $h(a)$ are as defined in Question 1.

Solution:

$$
\begin{aligned}
\frac{d^2z}{dx^2} &= \frac{d}{dx}\left(2\cos x - 3\sec^2 x\right) \\
&= -2\sin x - 6\sec x\frac{d}{dx}\sec x \\
&= -2\sin x - 6\sec x\sec x\tan x \\
&= -2\sin x - 6\sec^2 x\tan x
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
h''(a) &= \frac{d}{da}\left(2a\cos a - (a^2+5)\sin a\right) \\
&= (2\cos a - 2a\sin a) - (2a\sin a + (a^2+5)\cos a) \\
&= 2\cos a - 2a\sin a - 2a\sin a - a^2\cos a - 5\cos a \\
&= -(a^2+3)\cos a - 4a\sin a
\end{aligned} \tag{2}
$$

Question 3. Find the general antiderivatives of the following functions.

1. $f(t) = 2t^3 - 6t + 1$
2. $g(x) = \frac{3}{x^2}$
3. $r(\Theta) = 5 + \cos\Theta$

Solution:

$$
\int 2t^3 - 6t + 1\,dt = \frac{t^4}{2} - 3t^2 + t + C \tag{1}
$$

$$\begin{aligned}\int \frac{3}{x^2}\,dx &= \int 3x^{-2}/, dx \qquad (2)\\ &= -3x^{-1}+C\end{aligned}$$

$$\int 5+\cos\Theta\,d\Theta = 5\Theta+\sin\Theta+C \qquad (3)$$

Question 4. You may use a calculator on both parts of this question.

Part A. Imagine an elliptical oil slick that at some moment in time is 3 meters long and 2 meters wide. At this instant, it's length is increasing at a rate of 2 centimeters per minute, and its width at a rate of 1 centimeter per minute. How fast is the oil slick's area increasing?

Helpful information: The area of an ellipse is given by the equation $A=\pi ab$, where A is area, and a and b are the lengths of the semi-major and semi-minor axes respectively. What does that mean? The long axis of an ellipse is called its *major axis*; the short axis is called the *minor axis*. Half of these axes, i.e., the lines from the center of the ellipse to its edge in the longest and shortest dimensions, are the *semi-major* and *semi-minor* axes.

Solution:

$$\begin{aligned}\frac{dA}{dt} &= \pi(\frac{da}{dt}b+\frac{db}{dt}a)\\ &= \pi(0.01\text{ m/min}\times 1\text{ m}+0.005\text{ m/min}\times 1.5\text{ m})\\ &= 0.025\pi\text{ m}^2/\text{min}\end{aligned}$$

Part B. If the oil slick is a constant 1 millimeter thick, how fast (in units of volume per minute, e.g., cubic centimeters per minute) is oil entering the slick at the moment described in Part A?

More helpful information: The volume of such an oil slick is simply its thickness times its area.

Solution:

$$\begin{aligned}\frac{dV}{dt} &= \frac{d}{dt}(0.001\text{ m}\times A)\\ &= 0.001\text{ m}\times\frac{dA}{dt}\\ &= 0.001\text{ m}\times 0.025\pi\text{ m}^2/\text{min}\\ &= 0.000025\pi\text{ m}^3/\text{min}\\ &= 25\pi\text{ cm}^3/\text{min}\end{aligned}$$

Question 5. Find $\frac{dy}{dx}$ given the following definitions of y. These problems come from or at least are inspired by OpenStax *Calculus, Volume 1*:

1. $y=x^2\cos^4 x$ (exercise 234 in section 3.6).
2. $y=(6+\sec(\pi x^2))^2$ (inspired by exercise 236 in section 3.6).

Solution:

$$\begin{aligned}\frac{dy}{dx} &= 2x\cos^4 x + x^2\frac{d}{dx}\left(\cos^4 x\right) \qquad (1)\\ &= 2x\cos^4 x + x^2(4\cos^3 x(-\sin x))\\ &= 2x\cos^4 x - 4x^2\cos^3 x\sin x\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= 2(6+\sec(\pi x^2))\frac{d}{dx}\left(6+\sec(\pi x^2)\right) \qquad (2)\\ &= 2(6+\sec(\pi x^2))(\sec(\pi x^2)\tan(\pi x^2))\frac{d}{dx}\left(\pi x^2\right)\\ &= 2(6+\sec(\pi x^2))(\sec(\pi x^2)\tan(\pi x^2))2\pi x\\ &= 4\pi x(6+\sec(\pi x^2))\sec(\pi x^2)\tan(\pi x^2)\end{aligned}$$

Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the “Grade By” date above.