Math 221 06
Prof. Doug Baldwin

Problem Set 2 — More Limits

Complete by **Thursday, September 12**
Grade by **Tuesday, September 17**

Purpose

This problem set develops your ability to use algebraic simplifications along with limit laws to evaluate limits, as well as to evaluate limits with Mathematica. In particular, by the time you finish this problem set you should be able to . . .

- Use algebraic simplifications in evaluating limits
- Use limit laws to evaluate limits
- Use Mathematica to find limits.

Background

This problem set uses ideas from section 2.3 of our textbook. We discussed this material in classes between September 6 and 9.

This problem set also asks you to find limits using Mathematica. I plan to demonstrate how to do this in class on September 11, but if you want to try it sooner (or if I don't manage to do the demonstration), here is the main idea: Mathematica has a "Limit" function that finds limits. Its arguments are the expression whose limit you want to find, and the place at which you want that limit. For example, here is how I made Mathematica evaluate some limits we did by hand in class:

```
In[1]:= Limit[2 x + 3, x → 1]
Out[1]= 5

In[2]:= Limit[(x + 1)^2/Sqrt[x + 2], x → 0]
Out[2]= 1/√2

In[3]:= Limit[(x^2 - x - 2)/(x - 2), x → 2]
Out[3]= 3
```

Type the arrow indicating the place at which to find the limit by typing a hyphen ("-") immediately followed by a greater-than sign (">") on your keyboard; Mathematica will replace those two characters with the arrow shown in the examples.

Activity

Solve the following problems:

Question 1. Use algebra and limit laws to find the following limits. Once you have found the limits by hand, use Mathematica to confirm your answers:

1. $$\lim_{x \to 2} \frac{x-2}{x^2-2x}$$

   (Exercise 94 in section 2.3 of the textbook).

2. $$\lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$

   (Inspired by Exercise 98 in section 2.3 of the textbook).

3. $$\lim_{x \to \frac{\pi}{4}} \frac{\cos(2x)}{\cos x - \sin x}$$

   (Hint: review your trigonometric formulas — there's a list of them in appendix C.3 in the textbook. You can also assume — we'll see proofs in a couple of weeks — that for all real numbers a, $\lim_{x \to a} \sin x = \sin a$ and $\lim_{x \to a} \cos x = \cos a$. For checking your answer with Mathematica, the sine and cosine functions in Mathematica are "Sin" and "Cos" respectively – note the capital first letters, and remember that, as with all functions, arguments go in square brackets; you can enter the constant π into Mathematica by typing "Pi" – again, notice the capital letter.)

4. $$\lim_{x \to -3} \frac{\sqrt{x+4}-1}{x+3}$$

   (Exercise 102 from section 2.3 of the textbook).

Solution:

$$\begin{aligned}
\lim_{x\to 2}\frac{x-2}{x^2-2x} &= \lim_{x\to 2}\frac{x-2}{x(x-2)}\\
&= \lim_{x\to 2}\frac{1}{x}\\
&= \frac{1}{2}\\
\lim_{h\to 0}\frac{\frac{1}{2+h}-\frac{1}{2}}{h} &= \lim_{h\to 0}\frac{\frac{2}{2(2+h)}-\frac{2+h}{2(2+h)}}{h}\\
&= \lim_{h\to 0}\frac{\frac{2-2-h}{2(2+h)}}{h}\\
&= \lim_{h\to 0}\frac{\frac{-h}{2(2+h)}}{h}\\
&= \lim_{h\to 0}\frac{-1}{2(2+h)}\\
&= \frac{-1}{2(2+0)}\\
&= \frac{-1}{4}\\
\lim_{x\to\frac{\pi}{4}}\frac{\cos(2x)}{\cos x-\sin x} &= \lim_{x\to\frac{\pi}{4}}\frac{\cos^2 x-\sin^2 x}{\cos x-\sin x}\\
&= \lim_{x\to\frac{\pi}{4}}\frac{(\cos x-\sin x)(\cos x+\sin x)}{\cos x-\sin x}\\
&= \lim_{x\to\frac{\pi}{4}}\cos x+\sin x\\
&= \cos\frac{\pi}{4}+\sin\frac{\pi}{4}\\
&= \sqrt{2}\\
\lim_{x\to -3}\frac{\sqrt{x+4}-1}{x+3} &= \lim_{x\to -3}\frac{(\sqrt{x+4}-1)(\sqrt{x+4}+1)}{(x+3)(\sqrt{x+4}+1)}\\
&= \lim_{x\to -3}\frac{(x+4)-1}{(x+3)(\sqrt{x+4}+1)}\\
&= \lim_{x\to -3}\frac{x+3}{(x+3)(\sqrt{x+4}+1)}\\
&= \lim_{x\to -3}\frac{1}{\sqrt{x+4}+1}\\
&= \frac{1}{2}
\end{aligned}$$

Question 2. Consider the function

$$f(t)=\frac{(1+2t)^2-1}{t}$$

Part A. Give an example of a value a for which $f(a)$ is undefined, but $\lim_{t\to a} f(t)$ is defined. What is the value of the limit?

Solution: $f(0)$ is undefined because of the t in the denominator in the definition of f, but

$$\begin{aligned}
\lim_{t\to 0} f(t) &= \lim_{t\to 0} \frac{(1+2t)^2-1}{t} \\
&= \lim_{t\to 0} \frac{(1+4t+4t^2)-1}{t} \\
&= \lim_{t\to 0} \frac{4t+4t^2}{t} \\
&= \lim_{t\to 0}(4+4t) \\
&= 4
\end{aligned}$$

Part B. Give an example of a value b for which $f(b)$ is defined and $\lim_{t\to b} f(t) = f(b)$.

Solution: $f(b)$ is defined for all $b \neq 0$, for example $f(1) = \frac{(1+2\times 1)^2-1}{1} = 8$. Using the quotient, power, sum, difference, and basic limit laws shows that $\lim_{t\to 1} f(t)$ also equals 8.

Follow-Up

I will grade this exercise in a face-to-face meeting with you. During this meeting I will look at your solution, ask you any questions I have about it, answer questions you have, etc. Please bring a written solution to the exercise to your meeting, as that will speed the process along. Be sure to also bring either a print-out of your Mathematica work, or a computer with your work loaded into Mathematica.

Sign up for a meeting via Google calendar. Please make the meeting 15 minutes long, and schedule it to finish before the end of the "Grade By" date above.