Math 221 03 — Final Exam

May 16, 2019

General Directions. This is an open-book, open-notes, open-computer test. However, you may not communicate with any person, except me, during the test. You have the full exam period (200 minutes) in which to do the test. Put your answer to each question in the space provided (use the backs of pages if you need more space). Be sure to *show your work!* In situations where you use Mathematica or other technology to calculate part of an answer, showing your work includes identifying the places you used technology and showing or describing the commands/functions you used. I give partial credit for incorrect answers if you show correct steps leading up to them; conversely, I do not give full credit even for correct answers if it is not clear that you understand where those answers come from. Good luck.

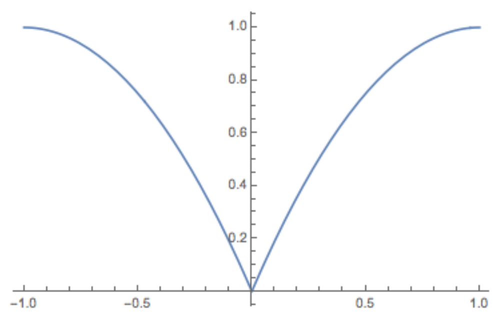
This exam contains 9 questions on 8 pages.

Question 1. (10 points) Phineas Phoole claims to know of a function $g(x)$ that has the following properties:

1. $g(0)$ is a local minimum
2. $g'(0)$ is undefined
3. $g''(x) < 0$ for $x < 0$
4. $g''(x) < 0$ for $x > 0$

Either sketch a graph of such a $g(x)$ over the interval $-1 \leq x \leq 1$, or explain why no such function can exist.

Solution: Such a function can exist, for example



Question 2. (15 points) Consider finding the antiderivative

$$\int 4\sin(2x)\cos(2x)\,dx$$

There are at least two substitutions you could use to find this antiderivative. What are they? Show each by giving its "u" and "du" parts, and then evaluate the antiderivative using each.

For up to 2 points of extra credit, explain why the two apparently different antiderivatives you get in answering this question both really are antiderivatives of $4\sin(2x)\cos(2x)$.

Solution: One substitution is $u = \sin(2x), du = 2\cos(2x)\,dx$. The antiderivative then becomes

$$\begin{aligned}\int 4\sin(2x)\cos(2x)\,dx &= \int 2u\,du \\ &= u^2 + C \\ &= \sin^2(2x) + C\end{aligned}$$

Another substitution is $u = \cos(2x), du = -2\sin(2x)\,dx$. The antiderivative now becomes

$$\begin{aligned}\int 4\sin(2x)\cos(2x)\,dx &= \int -2u\,du \\ &= -u^2 + K \\ &= -\cos^2(2x) + K\end{aligned}$$

These antiderivatives are equivalent because $\sin^2(2x) = 1 - \cos^2(2x)$, and the 1 can be absorbed into a constant of integration, leaving $\sin^2(2x) + C = -\cos^2(2x) + K$.

Question 3. (10 points) Find the value of b such that

$$\int_1^b 6x^2\,dx = 14$$

Solution:

$$\begin{aligned}\int_1^b 6x^2\,dx &= \left[2x^3\right]_1^b \\ &= 2b^3 - 2\end{aligned}$$

Thus we have

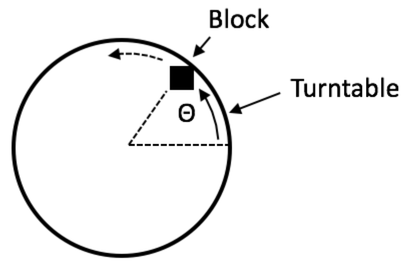
$$\begin{aligned}2b^3 - 2 &= 14 \\ 2b^3 &= 16 \\ b^3 &= 8 \\ b &= 2\end{aligned}$$

Question 4. (15 points) Imagine water flowing into a reservoir at a variable rate $w(t)$ gallons per minute, where t represents time. Assume the reservoir is initially empty, and that no water flows out of it. Explain how you could use a Riemann sum to model the volume of water in the reservoir.

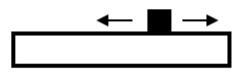
Solution: Divide the time interval of interest into n equal-sized sub-intervals, over which $w(t)$ can be assumed to be roughly constant. The total volume of water in the reservoir is then the sum of the amounts that flow in during each sub-interval. Since the amount of water flowing in during a sub-interval is the rate during that interval times the length of the interval, this sum is the Riemann sum...

$$W = \sum_{i=1}^{n} w(t_i)\Delta t$$

Question 5. (15 points) Imagine a small block sitting on a rotating disk (e.g., on a lazy Susan or turntable). Seen from above, the block moves in a circle; after moving for t seconds, it has rotated to angle Θ radians from its original position, as shown in Part A in this picture:



A



B

However, as seen from the side the block simply seems to move back and forth, as seen in Part B of the picture. The block's apparent position left or right of the center of the disk is proportional to $\cos\Theta$.

Assume that such a disk and block system is set up so that the block as viewed from the side has position $x = 4\cos\Theta$ inches from the center of the disk — positive distances are to the right of the center, negative to the left. Furthermore, suppose that after turning for t seconds, the turntable has rotated a total of $\Theta = 6t$ radians. Find a function of t that tells you how fast the viewer from the side sees the block moving back and forth.

Solution: Since speed is the derivative of position, we are looking for $\frac{dx}{dt}$, where x is the block's position left or right of the center. Furthermore, we know that $x = 4\cos\Theta$, and that $\Theta = 6t$. Plugging the equation for Θ into the one for x yields $x = 4\cos(6t)$, so $\frac{dx}{dt} = -4\sin(6t) \times 6 = -24\sin(6t)$ inches per second.

Question 6. (15 points) The designers of a hypothetical Mall of Geneseo have decided to put an arched window at one end of their mall. The upper edge of the window will follow the curve $y = 16 - x^4$, where y is height in feet. Assuming the bottom of the window is level at $y = 0$, what is the total area of the window?

Solution: The window fills the area under the curve, between points where $y = 0$. Solving $16 - x^4 = 0$ for x we get $x^4 = 16$ or $x = \pm 2$. The area is then

$$\begin{aligned} A &= \int_{-2}^{2} 16 - x^4 \, dt \\ &= \left[16x - \frac{x^5}{5} \right]_{-2}^{2} \\ &= (32 - \frac{32}{5}) - (-32 - \frac{-32}{5}) \\ &= \frac{128}{5} - \frac{-128}{5} \\ &= \frac{256}{5} \text{ sq ft} \end{aligned}$$

Question 7. (15 points) Imagine a square that is expanding in such a way that its area increases by 2 square meters per second. How fast is the length of its diagonal increasing when the area is 8 square meters?

Solution: If the area of the square is A square meters, then its sides are $\sqrt{A}$ meters long, and its diagonal is

$$\begin{aligned} L &= \sqrt{\sqrt{A}^2 + \sqrt{A}^2} \\ &= \sqrt{2A} \end{aligned}$$

meters. The rate at which the diagonal changes is

$$\begin{aligned} \frac{dL}{dt} &= \frac{1}{2\sqrt{2A}} 2 \frac{dA}{dt} \\ &= \frac{1}{\sqrt{2A}} \frac{dA}{dt} \\ &= \frac{2}{\sqrt{2A}} \frac{\text{m}}{\text{sec}} \end{aligned}$$

Plugging in $A = 8$ yields $\frac{dL}{dt} = \frac{2}{\sqrt{16}} = \frac{1}{2} \frac{\text{m}}{\text{sec}}$.

Question 8. (10 points) Izzy Newton, Sir Isaac's umpteenth descendant, has yet another of his historic but damaged notes. This one deals with limits, and reads...

We can find

$$\lim_{x\to\infty} \frac{5x^2}{2x^2+x-7}$$

by noticing that

$$\begin{aligned} \lim_{x\to\infty} \frac{5x^2}{2x^2+x-7} &= ??? \\ &= \lim_{x\to\infty} \frac{5}{2+\frac{1}{x}-\frac{7}{x^2}} \\ &= \frac{5}{2} \end{aligned}$$

Fill in an expression that could plausibly go in the missing part (i.e., the "???") of this example.

Solution: Isaac probably factored x^2 out of numerator and denominator:

$$\begin{aligned} \lim_{x\to\infty} \frac{5x^2}{2x^2+x-7} &= \lim_{x\to\infty} \frac{5x^2}{x^2(2+\frac{1}{x}-\frac{7}{x^2})} \\ &= \lim_{x\to\infty} \frac{5}{2+\frac{1}{x}-\frac{7}{x^2}} \\ &= \frac{5}{2} \end{aligned}$$

Question 9. (15 points) A totally fictitious psychologist claims that the average college professor's daytime alertness is described by the equation

$$A = \frac{3t}{t^2 + 4}$$

where A is alertness (in "lerts," an utterly made-up unit) and t is the number of hours since the professor got out of bed in the morning. Find the maximum and minimum alertness, in lerts, of the average professor during a 12-hour day (i.e., over the interval $[0, 12]$).

Solution: Start by finding critical points of the alertness function. To do this, we need the derviative of A:

$$\begin{aligned} \frac{dA}{dt} &= \frac{3(t^2+4) - (3t)(2t)}{(t^2+4)^2} \\ &= \frac{3t^2 + 12 - 6t^2}{(t^2+4)^2} \\ &= \frac{12 - 3t^2}{(t^2+4)^2} \end{aligned}$$

Since this is defined everywhere, the critical points will be where it equals 0:

$$\begin{aligned} \frac{12 - 3t^2}{(t^2+4)^2} &= 0 \\ 12 - 3t^2 &= 0 \\ 3t^2 &= 12 \\ t^2 &= 4 \\ t &= \pm 2 \end{aligned}$$

Since -2 is outside the interval $[0, 12]$, we only need to consider $t = 2$ as a critical point.

Minimum and maximum alertness will occur at the critical point or at one of the interval's endpoints, so we compare all 3:

t	A
0	0
2	$\frac{6}{8} = \frac{3}{4}$
12	$\frac{36}{148} = \frac{9}{37}$

The minimum alertness is thus 0 lerts when the professor gets out of bed, and the maximum is $\frac{3}{4}$ lerts 2 hours later.

Have a Good Summer!