Math 221 03 — Hour Exam 2

April 11, 2019

These are some questions from recent tests I've given, covering material that will be on our second hour exam. There are more questions here than I used on any one exam. I have included the original point values of the questions, to give you some sense of what I consider relatively "big" versus relatively "small" questions. I've also included solutions to all the questions, although the questions will be most helpful for studying if you answer them on your own before looking at my solutions. And feel free to ask questions about any of these, of course.

Question 1. (5 points) Phineas Phoole has discovered a new differentiation rule, which he calls "the sine rule." The sine rule says that if $f(x)$ is a function, then the derivative of $\sin(f(x))$ is $f'(x)\cos(f(x))$. Phineas is very proud of this new rule, but unfortunately can't give a proof of it. Give the missing proof, i.e., explain why Phineas's sine rule is correct.

> **Solution:** The sine rule is just a direct application of the chain rule.

Question 2. (10 points) Imagine that f is a function, but that all you know about it is that $f(100) = 10$ and $f'(100) = 0.5$. Estimate the value of $f(99)$.

Solution: $f(99)$ can be estimated as

$$\begin{aligned} f(99) &\approx f(100) + f'(100)(99 - 100) \\ &= 10 + 0.5 \times -1 \\ &= 9.5 \end{aligned}$$

Question 3. (15 points) The market analysts at Geneseo Widget Works predict that over the next 10 months the monthly demand for widgets will be

$$w = 955 - 5t^2 + 30t$$

widgets per month, where t is the number of months into the future (so $0 \le t \le 10$).

Find the absolute minimum and maximum monthly demand for widgets over the next 10 months.

Solution: Find critical points in the demand function:

$$\frac{dw}{dt} = -10t + 30$$

Setting this equal to 0 and solving for t yields

$$\begin{aligned} \frac{dw}{dt} &= 0 \\ -10t + 30 &= 0 \\ 10t &= 30 \\ t &= 3 \end{aligned}$$

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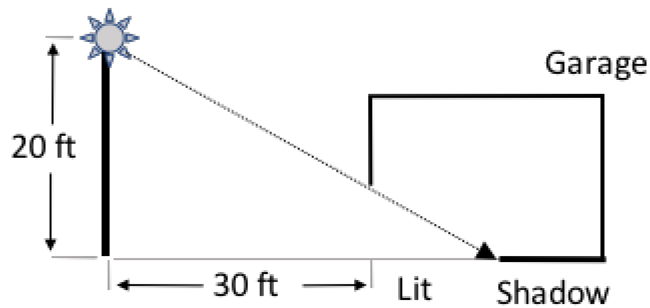
The maximum and minimum will occur at this critical point or at the endpoints of the interval:

- $w(0) = 955 - 5 \times 0^2 + 30 \times 0 = 955$
- $w(3) = 955 - 5 \times 3^2 + 30 \times 3 = 955 - 45 + 90 = 1000$
- $w(10) = 955 - 5 \times 10^2 + 30 \times 10 = 955 - 500 + 300 = 755$

The smallest of these is 755 widgets at $t = 10$, and the largest is 1000 at $t = 3$.

Question 4. (15 points) Imagine a garage with a roll-up door, as seen below. When the door is rolled up, light from a street light across the street shines through it and lights up part of the garage floor. How much of the floor is lit depends on how far up the door is rolled. Suppose the street light is 30 feet from the door, and 20 feet tall.



At a particular instant, the door is opening at a rate of $\frac{1}{2}$ foot per second, and has an opening 10 feet high. How fast is the shadow area on the floor receding into the garage?

Solution: Let the length of the lit part of the garage be x, and the height of the door opening be h. We then have 2 similar triangles, one involving the door (height h feet) and lit section of floor (length x feet), and the other involving the street light (height 20 feet) and the lit floor plus the street (length $30 + x$ feet). Therefore

$$\begin{aligned} \frac{x}{h} &= \frac{30+x}{20} \\ 20x &= 30h + hx \\ 20x - hx &= 30h \\ x(20-h) &= 30h \\ x &= \frac{30h}{20-h} \end{aligned}$$

Thus

$$\begin{aligned} \frac{dx}{dt} &= \frac{30(20-h)+30h}{(20-h)^2}\frac{dh}{dt} \\ &= \frac{600-30h+30h}{(20-h)^2}\frac{dh}{dt} \\ &= \frac{600}{(20-h)^2}\frac{dh}{dt} \end{aligned}$$

Plugging in the numbers given in the question yields

$$\begin{aligned} \frac{dx}{dt} &= \frac{600}{(20-h)^2}\frac{dh}{dt} \\ &= \frac{600}{(20-10)^2}\frac{1}{2} \\ &= \frac{600}{2\times 10^2} \\ &= \frac{600}{200} \\ &= 3 \quad \text{feet/second} \end{aligned}$$

Question 5. (10 points) Izzy Newton (Isaac Newton's descendant with a trove of his old and damaged notebooks, who you might remember from the first exam) has found another of Isaac's calculus examples. In this example, Isaac writes...

> Suppose that $x \sin y = y \cos x$. Then we find $\frac{dy}{dx}$ as follows:
> ???
> And so
> $$\frac{dy}{dx} = \frac{\sin y + y \sin x}{\cos x - x \cos y}$$

Give calculations that could plausibly have gone in the damaged section (marked with "???") of this example.

Solution: Isaac probably used implicit differentiation:

$$\begin{aligned}
\frac{d}{dx}(x \sin y) &= \frac{d}{dx}(y \cos x) \\
\sin y + x \cos y \frac{dy}{dx} &= \frac{dy}{dx} \cos x - y \sin x \\
\sin y + y \sin x &= \cos x \frac{dy}{dx} - x \cos y \frac{dy}{dx} \\
\sin y + y \sin x &= (\cos x - x \cos y)\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{\sin y + y \sin x}{\cos x - x \cos y}
\end{aligned}$$