Math 221, Prof. Baldwin — Sample Hour Exam 1

October, 2019

General Directions. The following are questions from some of my recent Math 221 hour exams on limits and derivatives. There are not necessarily the same number of questions as on any actual exam, since I'm more interested in giving you a variety of things to practice on than in recreating an actual exam. I have, however, kept the point values for the questions, so you can have some sense of how hard I considered the questions to be.

Question 1. (5 points) Give an example of a function $f(x)$ whose derivative is $9x^2 - 2x$.

Solution: Any antiderivative will do, so in general

$$\begin{aligned} f(x) &= \int 9x^2 - 2x\,dx \\ &= 3x^3 - x^2 + C \end{aligned}$$

Question 2. (15 points) If $f(x)$ is a function, its "symmetric difference quotient" is a fraction of the form

$$\frac{f(x+h) - f(x-h)}{2h}$$

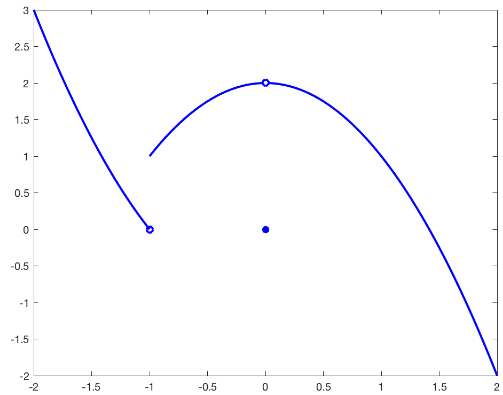
where h is a number, typically small. Show that for $f(x) = x^2 - 3$, the limit of f's symmetric difference quotient as h approches 0 equals f's derivative[1].

Solution:

$$\begin{aligned} \lim_{h\to 0} \frac{((x+h)^2 - 3) - ((x-h)^2 - 3)}{2h} &= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2 - 3) - (x^2 - 2xh + h^2 - 3)}{2h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 3 - x^2 + 2xh - h^2 + 3}{2h} \\ &= \lim_{h\to 0} \frac{4xh}{2h} \\ &= \lim_{h\to 0} 2x \\ &= 2x \\ &= f'(x) \end{aligned}$$

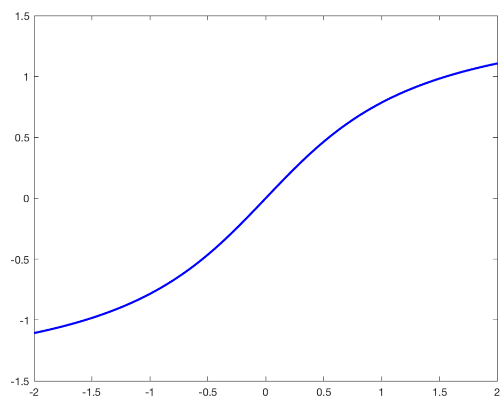
[1] People really do use symmetic difference quotients to estimate derivatives, although you have to be careful doing so because there are some functions for which the limit of the symmetric difference quotient is not equal to the derivative.

Question 3. (10 points) Here is a graph of a function $g(x)$. Identify all the numbers a in the open interval $(-2, 2)$ where $\lim_{x \to a} g(x)$ does not exist. For each such a, explain in about one sentence why the limit doesn't exist.



Solution: The only place where the limit doesn't exist is at $a = -1$, where the limits from the left and right differ.

Question 4. (5 points) Here is a graph of a certain function near $x = 0$:



At roughly which of the x values in the graph is this function's derivative largest? Explain your choice in about one sentence.

Solution: The derivative is largest where the slope of the graph is greatest, i.e., where the graph is rising fastest. This seems to happen around $x = 0$.

Question 5. (10 points) A pump pumps water alternately into and out of a bucket in such a manner that t minutes after the pump starts there are $2\sin(\pi t) + 2$ gallons of water in the bucket. What is the rate (in gallons/minute) at which the pump moves water into or out of the bucket, as a function of t?

Solution: The rate of inflow/outflow is the derivative of W, the amount of water in the bucket. Use the derivative of sine, the chain rule, and simpler differentiation rules to find...

$$\begin{aligned} \frac{dW}{dt} &= 2\cos(\pi t) \times \pi \\ &= 2\pi \cos(\pi t) \text{ gal/min} \end{aligned}$$

Question 6. Izzy Newton, great-to-the-umpteenth grandchild of Isaac, the co-discoverer of calculus, has found an old trunk in the attic filled with Sir Isaac's long-lost notes. Izzy translates the notes into modern language and notation, but finds many places where age has made the notes illegible, marked by "???" in the examples below.

Part A. (10 points) In one of the notes, Isaac shows how he calculates a certain derivative:

Let $f(x)$ be

$$f(x) = \frac{2x^3 - 3x^2}{x^2 - 2x}$$

Then

$$\begin{aligned} f'(x) &= ??? \\ &= \frac{(6x^4 - 12x^3 - 6x^3 + 12x^2) - (4x^4 - 4x^3 - 6x^3 + 6x^2)}{(x^2 - 2x)^2} \\ &= \frac{2x^4 - 8x^3 + 6x^2}{(x^2 - 2x)^2} \end{aligned}$$

Identify a differentiation rule Isaac could have used to get $f'(x)$ in the damaged step, and give an expression based on that rule that could be the missing part of his example.

Solution: Isaac probably used the quotient rule, in which case the damaged equation could be

$$f'(x) = \frac{(6x^2 - 6x)(x^2 - 2x) - (2x^3 - 3x^2)(2x - 2)}{(x^2 - 2x)^2}$$

Part B. (5 points) In another note, Isaac writes...

Let $f(t)$ be

$$f(t) = ???$$

Then

$$f'(t) = 6t^2 - 5$$

What could the missing definition of $f(t)$ plausibly be? Explain your reasoning in a sentence or two.

Solution:

$$\begin{aligned} f(t) &= \int 6t^2 - 5\, dt \\ &= 2t^3 - 5t + C \end{aligned}$$