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Patchy solution of a Francis–Byrnes–Isidori partial differential equation

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SUMMARY

The solution to the nonlinear output regulation problem requires one to solve a first-order partial differential equation, known as the Francis–Byrnes–Isidori equations. In this paper, we propose a method to compute approximate solutions to the Francis–Byrnes–Isidori equations when the zero dynamics of the plant are hyperbolic and the exosystem is two dimensional. With our method, we are able to produce approximations that converge uniformly to the true solution. Our method relies on the periodic nature of two-dimensional analytic center manifolds. Copyright © 2012 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Consider the control system

$$\begin{aligned} \dot{x} &= f(x,u,w) \\ \dot{w} &= s(w) \\ y &= h(x,u,w), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control variable, $w \in \mathbb{R}^q$ is an exogenous variable, and $y \in \mathbb{R}^p$ is the output variable. The maps $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^n$, $s : \mathbb{R}^q \to \mathbb{R}^q$, and $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^p$ are all assumed to be sufficiently smooth and satisfy $f(0,0,0) = 0$, $s(0) = 0$, and $h(0,0,0) = 0$. The variable w represents a disturbance and/or a reference signal, and its dynamics are commonly referred to as the *exosystem*. The *state feedback regulator problem* [1] is to find a state feedback control $u = \alpha(x,w)$, with $\alpha(0,0) = 0$, such that the equilibrium $x = 0$ of the dynamical system

$$\dot{x} = f(x, \alpha(x,0), 0)$$

is exponentially stable and that for each sufficiently small initial condition (x_0, w_0), the solution of (1) with $u = \alpha(x,w)$ satisfies

$$\lim_{t\to\infty} y(t) = 0.$$

A characterization of the state feedback regulator problem for linear systems was given by Francis [2] and later generalized to nonlinear systems by Isidori and Byrnes [1]. As shown in [1],

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the solvability of the regulator problem can be reduced to the solvability of a system of partial differential equations (PDEs), which in the linear case reduce to the Sylvester type equation obtained by Francis. For this reason, we refer to these equations as the Francis–Byrnes–Isidori (FBI) PDEs. For completeness, we state the main result of [1] (for the definition of Poisson stability used, see Remark 1.1).

Theorem 1.1 ([1]*)*
Assume that the equilibrium $w = 0$ of the exosystem is Lyapunov stable and there is a neighborhood of $w = 0$ in which every point is Poisson stable. Assume further that the pair

$$\left(\frac{\partial f}{\partial x}(0,0,0), \frac{\partial f}{\partial u}(0,0,0)\right) \tag{2}$$

is stabilizable. Then the state feedback regulator problem is solvable if and only if there exists C^k $(k \geqslant 2)$ mappings $\pi : \Omega \to \mathbb{R}^n$, with $\pi(0) = 0$, and $\kappa : \Omega \to \mathbb{R}^m$, with $\kappa(0) = 0$, both defined in a neighborhood $\Omega \subseteq \mathbb{R}^q$ of $w = 0$ and satisfying

$$\begin{aligned} \frac{\partial \pi}{\partial w}(w)s(w) &= f(\pi(w), \kappa(w), w) \\ 0 &= h(\pi(w), \kappa(w), w). \end{aligned} \tag{3}$$

Given a solution pair (π, κ) to the FBI equations (3), a state feedback solving the regulator problem is given by

$$\alpha(x, w) = \kappa(w) + K(x - \pi(w)),$$

where $K \in \mathbb{R}^{m \times n}$ is any feedback matrix rendering the pair (2) asymptotically stable. In general, solutions to the FBI equations, being singular quasilinear PDEs with constraints, may not exist. However, for a class of control-affine systems, it is shown in [1] that the solvability of the FBI equations is a property of the zero dynamics of (1). Roughly speaking, if the zero dynamics of (1) has a hyperbolic equilibrium at the origin, then a solution to the FBI equations exists by the center manifold theorem [3]. It is known, however, that center manifolds suffer from a number of subtle properties associated with uniqueness and differentiability [4]. Despite these difficulties, a C^∞ dynamical system possess a C^k center manifold for each $k \geqslant 1$; moreover, it is possible to obtain approximate solutions of arbitrarily high-order via Taylor series [3]. In this respect, Huang and Rugh [5] and Krener [6] provide a method to compute approximate solutions to the FBI equations via Taylor polynomials, yielding approximate output regulation. Naturally, if one desires a more accurate output tracking controller, then one can increase the degree of the Taylor approximation; however, there are two main drawbacks in doing so. First, increasing the order increases the accuracy of the approximation but on a possibly *smaller* domain because of the rapidly growing behavior of high-order polynomials away from the origin. Second, the number of monomials in q variables of degree N is

$$\binom{q + N - 1}{N},$$

a number growing rapidly in N.

In this paper, we present a method to compute solutions to the FBI equations for the class of real analytic SISO control-affine systems

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ \dot{w} &= s(w) \\ y &= h(x) + p(w), \end{aligned} \tag{4}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, $s : \mathbb{R}^q \to \mathbb{R}^q$, $h : \mathbb{R}^n \to \mathbb{R}$, and $p : \mathbb{R}^q \to \mathbb{R}$ are real analytic about the origin. Furthermore, we will restrict our considerations to two-dimensional

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exosystems, that is, $q = 2$, whose linear part contains non-zero eigenvalues. Our method is based on the existence and uniqueness results for two-dimensional analytic center manifolds in [7] and on a high-order patchy method similar to [8]. A key strength of our approximation method, which relies on the periodic nature of the solution to the FBI equations for (4), is the reduction of the computational effort inherent in a direct Taylor polynomial approximation.

The organization of this paper is as follows. In Section 2, we briefly summarize the key insight provided in [1] on how the solvability of the FBI equations can be reduced to the problem of solving a center manifold equation provided that the zero dynamics of (4) are hyperbolic. With this simplification, we show how the standard stability assumptions on the exosystem lead to a direct application of the results in [7] to deduce uniqueness of solutions to the FBI equations of (4). In Section 3, we describe a continuation method to compute a high-order piecewise smooth approximation to the solution of the FBI equations and prove that a sequence of approximations generated by our method converges uniformly to the true solution. Finally, in Section 4, we illustrate our method on examples and then make some concluding remarks.

Remark 1.1
Henceforth, it will be implicitly assumed that the exosystem has an equilibrium $w = 0$ that is Lyapunov stable and that there is a neighborhood of $w = 0$ in which every point is Poisson stable. We will refer to this type of stability as *neutral stability*. By Poisson stability, we mean the following. An initial condition x_0 of the dynamical system $\dot{x} = f(x)$ is Poisson stable if the flow $\Phi_t^f(x_0)$ of the vector field f is defined for all $t \in \mathbb{R}$, and for each neighborhood U of x_0 and each real number $T > 0$, there exists a time $t_1 > T$ such that $\Phi_{t_1}^f(x_0) \in U$ and a time $t_2 < -T$ such that $\Phi_{t_2}^f(x_0) \in U$.

2. REAL ANALYTIC AND PERIODIC SOLUTIONS TO THE FRANCIS–BYRNES–ISIDORI EQUATIONS

As shown in [1], a key simplification in the problem of solving the FBI equations for a system of the form (4) consists in reducing it to the problem of solving a center manifold equation for the zero dynamics of (4). Following [1] and using the now standard notation in [9], assume that the triple $\{f, g, h\}$ from (4) has relative degree $1 \leqslant r < n$ at $x = 0$ and let (z, ξ) denote the standard normal coordinates, where $\xi = (h(x), L_f h(x), \ldots, L_f^{r-1} h(x))$ and z is such that $L_g z = 0$. In the (z, ξ) coordinates, (4) takes the form

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \\ \dot{\xi}_1 &= \xi_2, \ldots, \dot{\xi}_{r-1} = \xi_r \\ \dot{\xi}_r &= b(z, \xi) + a(z, \xi)u \\ \dot{w} &= s(w) \\ y &= \xi_1 + p(w). \end{aligned} \tag{5}$$

The zero dynamics of (4) are given by the dynamical system

$$\dot{z} = f_0(z, 0). \tag{6}$$

Define functions $\varphi_i : \mathbb{R}^q \to \mathbb{R}$ by $\varphi_i(w) = -L_s^{i-1} p(w)$, $1 \leqslant i \leqslant r$, set $\varphi(w) = (\varphi_1(w), \ldots, \varphi_r(w))$, and let

$$u_e(x, w) = -\frac{L_f^r h(x) + L_s^r p(w)}{L_g L_f^{r-1} h(x)}.$$

Then it is straightforward to verify that if ϕ satisfies the PDE

$$\frac{\partial \phi}{\partial w}(w) s(w) = f_0(\phi(w), \varphi(w)), \tag{7}$$

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then $\pi(w) := (\phi(w), \varphi(w))$ and $\kappa(w) := u_e(\pi(w), w)$ constitute a solution pair to the FBI equations of (5). If the origin of (6) is hyperbolic, that is, the matrix $\partial f_0/\partial z(0,0)$ has eigenvalues off the imaginary axis, then (7) is the equation that is satisfied by any center manifold $\{(z,w) : z = \phi(w)\}$ of the coupled dynamical system

$$\begin{aligned} \dot{z} &= f_0(z, \varphi(w)) \\ \dot{w} &= s(w). \end{aligned} \tag{8}$$

Hence, in the hyperbolic case, the problem of solving the FBI equations associated to the original system (4) is reduced to solving the center manifold equation associated to (8). Although this simplification is significant, solutions to center manifolds suffer from subtleties associated with uniqueness and differentiability [4]. For example, it is known that an analytic dynamical system does not generally posses an analytic center manifold, thereby forcing one to seek a center manifold solution that is only C^k $(k = 2, 3, \ldots)$ and thus not necessarily unique. As an example, the polynomial dynamical system

$$\begin{aligned} \dot{z} &= -z + w_1^2 + w_2^2 \\ \dot{w}_1 &= -w_2 - \frac{1}{2} w_1 (w_1^2 + w_2^2) \\ \dot{w}_2 &= w_1 - \frac{1}{2} w_2 (w_1^2 + w_2^2), \end{aligned}$$

which is of the form (8), has the property that each center manifold has Taylor series

$$\sum_{i=1}^{\infty} (i-1)!(w_1^2 + w_2^2)^i,$$

which has vanishing radius of convergence. Despite these difficulties, a special case for which sharp uniqueness and differentiability results exist is for two-dimensional center manifolds and is given by the following theorem by Aulbach [7].

Theorem 2.1 ([7]*)*
Consider the ordinary differential equation

$$\begin{aligned} \dot{z} &= Bz + Z(w_1, w_2, z) \\ \dot{w}_1 &= -w_2 + P(w_1, w_2, z) \\ \dot{w}_2 &= w_1 + Q(w_1, w_2, z), \end{aligned} \tag{9}$$

where w_1, $w_2 \in \mathbb{R}$, $z \in \mathbb{R}^n$, and P, Q, and Z are real analytic functions about the origin and have Taylor series beginning with quadratic terms. Suppose that the matrix B has no eigenvalues on the imaginary axis. If the local center manifold dynamics of (9) are Lyapunov stable and non-attractive, then (9) has a uniquely determined local center manifold that is analytic and generated by a family of periodic solutions.

Aulbach's result has a direct application to the output regulation problem, as given by the following theorem.

Theorem 2.2
Suppose that in (4) the exosystem is two dimensional and $\partial s/\partial w(0)$ has non-zero eigenvalues. Suppose that f, g, h, and p are real analytic mappings about $x = 0$ and $w = 0$, respectively, and that the triple $\{f, g, h\}$ has a well-defined relative degree $1 \leqslant r < n$ at $x = 0$. If the zero dynamics of (4) are hyperbolic, then there exist unique and real analytic mappings (π, κ) solving the associated FBI equations of (4).

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Proof

By assumption and neutral stability of the exosystem, the eigenvalues of the exosystem are non-zero and purely imaginary. Indeed, if the eigenvalues were not purely imaginary, then $w = 0$ would necessarily be either a repelling or an attractive equilibrium, contradicting the assumption of neutral stability. Now because $B := \partial f_0/\partial z(0,0)$ contains eigenvalues off the imaginary axis, there exists an analytic coordinate change [10] about the origin such that (8) takes the form

$$\begin{aligned} \dot{z} &= Bz + Z(w_1, w_2, z) \\ \dot{w}_1 &= -w_2 + P(w_1, w_2) \\ \dot{w}_2 &= w_1 + Q(w_1, w_2), \end{aligned} \tag{10}$$

where P, Q, and Z are analytic at the origin and have Taylor series beginning with quadratic terms. From (10), we can observe that the dynamics of any center manifold of (10) are equivalent to the exosystem dynamics, which by assumption are Lyapunov stable and non-attractive. Aulbach's theorem completes the proof. □

Remark 2.1

Theorem 2.2 actually holds for more general MIMO control-affine systems with $m = p$. In [11], it is shown that if the composite control-affine system

$$\begin{aligned} \dot{x} &= f(x, w) + \sum_{i=1}^{m} g_i(x, w) u_i \\ \dot{w} &= s(w) \\ y &= h(x, w) \end{aligned}$$

has a well-defined relative degree at $(x, w) = (0, 0)$, then the associated FBI equations are solvable if the zero dynamics of the composite system are hyperbolic. In this case, the FBI equations reduce to a center manifold equation of the form (7) so that Aulbach's theorem can be applied when the exosystem is two dimensional.

Example 2.1

The dynamics of a cart and inverted pendulum system can be written in the form.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{g}{\ell}\sin(x_3) - \frac{1}{\ell}\cos(x_3) u, \end{aligned} \tag{11}$$

where x_1 is the position of the cart, x_3 is the angle the pendulum makes with the vertical, g is the acceleration due to gravity, ℓ is the length of the rod, and u is the control force. With $h(x) = x_1$, the system has relative degree $r = 2$ at $x = 0$ and therefore $(\xi_1, \xi_2) = \xi(x) = (h(x), L_f h(x)) = (x_1, x_2)$. With $(z_1, z_2) = z(x) = (x_3, x_4 + (x_2/\ell)\cos(x_3))$, the zero dynamics are given by

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{g}{\ell}\sin(z_1), \end{aligned}$$

whose linearization at $z = 0$ has eigenvalues $\pm\sqrt{g/\ell}$. Hence, with system output $y = x_1 + p(w)$ (p real analytic) and a two-dimensional real analytic exosystem whose linearization has non-zero eigenvalues, there exists a unique and real analytic solution to the associated FBI equations of the cart and inverted pendulum system (11).

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3. COMPUTATION OF THE CENTER MANIFOLD

In this section, we outline a continuation method to compute the solution to the FBI equations in the case of two-dimensional exosystem and real analytic data. As described in the previous section, for the nonlinear control systems in consideration, the solvability of the FBI equations can be reduced to solving a center manifold equation for a dynamical system of the form

$$\begin{aligned} \dot{z} &= Bz + \bar{Z}(w_1, w_2, z) \\ \dot{w}_1 &= -w_2 + P(w_1, w_2) \\ \dot{w}_2 &= w_1 + Q(w_1, w_2), \end{aligned} \tag{12}$$

where $w = (w_1, w_2) \in \mathbb{R}^2$, $z \in \mathbb{R}^n$, $\bar{Z}$, P, and Q are real analytic mappings beginning with quadratic terms, and the eigenvalues of B have non-zero real parts. We will therefore limit our considerations to solving the center manifold equation for (12). It will be assumed that the w-dynamics have $w = 0$ as a Lyapunov stable and non-attractive equilibrium. By Theorem 2.1, there exists a unique analytic mapping $\phi(w_1, w_2)$, defined locally about $w = 0$, solving the center manifold PDE associated to (12).

Our method is best described on the representation of (12) in polar coordinates. Hence, we apply the transformation $(w_1, w_2, z) = (r\cos\theta, r\sin\theta, z)$ to (12) yielding a system of the form

$$\begin{aligned} \dot{r} &= r\hat{R}(\theta, r) \\ \dot{\theta} &= 1 + \hat{\Theta}(\theta, r) \\ \dot{z} &= Bz + \hat{Z}(\theta, r, z), \end{aligned} \tag{13}$$

where $\hat{R}$, $\hat{\Theta}$, $\hat{Z}$ are analytic functions converging for each $\theta \in [0, 2\pi]$ and $|r| \leqslant a$, $\|z\| \leqslant a$, where $a > 0$ is a positive constant. Define $\hat{f}(\theta, r, z) = Bz + \hat{Z}(\theta, r, z)$. The center manifold PDE for (13) is

$$\hat{f}(\theta, r, \psi(\theta, r)) = \frac{\partial \psi}{\partial \theta}[1 + \hat{\Theta}(\theta, r)] + \frac{\partial \psi}{\partial r} r\hat{R}(\theta, r) \tag{14}$$

for the unknown analytic mapping $\psi(\theta, r)$ $(= \phi(r\cos\theta, r\sin\theta))$. The mapping ψ has a power series representation

$$\psi(\theta, r) = \sum_{i=1}^{\infty} e_i(\theta)\frac{r^i}{i!}$$

converging in a cylinder of the form $\theta \in [0, 2\pi]$, $|r| \leqslant \epsilon$, and with 2π-periodic coefficients $e_i(\theta)$ [7]. By eliminating the time variable t, (13) can be reduced to

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = rR(\theta, r) \tag{15a}$$

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = Bz + Z(\theta, r, z). \tag{15b}$$

Define $f(\theta, r, z) = Bz + Z(\theta, r, z)$. From (14), it follows that

$$f(\theta, r, \psi(\theta, r)) = \frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}\theta}. \tag{16}$$

We now give a brief sketch of our method. Let $r(\theta)$ be a solution to (15a) and define the mapping

$$\Psi(\theta, \sigma) = \psi(\theta, r(\theta) + \sigma)$$

for $\theta \in [0, 2\pi]$ and $|\sigma|$ small. We note that with $r = r(\theta)$ substituted into the right-hand side of (15b), the curve $\Psi(\theta, 0) = \psi(\theta, r(\theta))$ is the solution to (15b) with initial condition $z(0) = \psi(0, r(0))$. For $|\sigma|$ sufficiently small, we have a power series representation

$$\Psi(\theta, \sigma) = \Psi(\theta, 0) + \sum_{i=1}^{\infty} \frac{\partial^i \Psi}{\partial \sigma^i}(\theta, 0)\frac{\sigma^i}{i!} \tag{17}$$

DOI: 10.1002/rnc

converging for all $\theta \in [0, 2\pi]$ and having 2π-periodic coefficients $\partial^i \Psi/\partial\sigma^i(\theta, 0)$. In fact, it is easy to see that

$$\frac{\partial^i \Psi}{\partial\sigma^i}(\theta, 0) = \frac{\partial^i \psi}{\partial r^i}(\theta, r(\theta)). \tag{18}$$

By construction, the mapping Ψ is a perturbation of $\psi(\theta, r(\theta))$ in the radial direction, the amount of perturbation given by the parameter σ. Our method is based on computing the Taylor series approximation

$$\Psi^N(\theta, \sigma) = \Psi(\theta, 0) + \sum_{i=1}^{N} \frac{\partial^i \Psi}{\partial\sigma^i}(\theta, 0)\frac{\sigma^i}{i!}$$

and using it to build the center manifold along $r(\theta)$ in the radial direction. Having followed Ψ^N along a small annular region, say of the form

$$\{(\theta, r) : 0 \leqslant \theta \leqslant 2\pi, r(\theta) \leqslant r < r(\theta) + \epsilon\},$$

we compute a new radial curve $\theta \mapsto \tilde{r}(\theta)$ with initial condition $\tilde{r}(0) = r(0) + \epsilon$, compute the new corresponding Taylor series approximation $\tilde{\Psi}^N$ for $\tilde{\Psi}(\theta, \sigma) = \psi(\theta, \tilde{r}(\theta) + \sigma)$, and then continue building the center manifold by following $\tilde{\Psi}^N$ along the annular region

$$\{(\theta, r) : 0 \leqslant \theta \leqslant 2\pi, \tilde{r}(\theta) \leqslant r < \tilde{r}(\theta) + \tilde{\epsilon}\}.$$

This process is repeated and the annular regions, along with the corresponding approximations, are patched together to form a piecewise smooth approximation to the true solution ψ.

3.1. Computation of the coefficients $\frac{\partial^i \Psi}{\partial\sigma^i}$

To compute the Taylor series approximation Ψ^N, it is necessary to compute the θ-dependent coefficients $\partial^i \Psi/\partial\sigma^i(\theta, 0)$ appearing in (17), which can be performed in the following way. From the definition of Ψ, a direct computation gives that

$$\frac{\partial\Psi}{\partial\theta} = \frac{\partial\psi}{\partial\theta}(\theta, r(\theta) + \sigma) + \frac{\partial\psi}{\partial r}(\theta, r(\theta) + \sigma)\frac{\mathrm{d}r}{\mathrm{d}\theta},$$

which when combined with (16) yields

$$\frac{\partial\Psi}{\partial\theta} = f(\theta, r(\theta) + \sigma, \Psi(\theta, \sigma)). \tag{19}$$

Using (19), we can derive a linear inhomogeneous ODE for the coefficient $\partial^i \Psi/\partial\sigma^i(\theta, 0)$. Indeed, differentiating (19) with respect to σ and interchanging the order of differentiation yields

$$\frac{\partial}{\partial\theta}\left(\frac{\partial\Psi}{\partial\sigma}(\theta, \sigma)\right) = \frac{\partial f}{\partial z}(\theta, r(\theta) + \sigma, \Psi(\theta, \sigma))\frac{\partial\Psi}{\partial\sigma}(\theta, \sigma) + \frac{\partial f}{\partial r}(\theta, r(\theta) + \sigma, \Psi(\theta, \sigma))$$

and therefore

$$\frac{\partial}{\partial\theta}\left(\frac{\partial\Psi}{\partial\sigma}(\theta, 0)\right) = A(\theta)\frac{\partial\Psi}{\partial\sigma}(\theta, 0) + \frac{\partial f}{\partial r}(\theta, r(\theta), \Psi(\theta, 0)),$$

where the matrix $A(\theta) = \partial f/\partial z(\theta, r(\theta), \Psi(\theta, 0))$. In general, it can be verified by induction that

$$\frac{\partial}{\partial\theta}\left(\frac{\partial^i \Psi}{\partial\sigma^i}(\theta, 0)\right) = A(\theta)\frac{\partial^i \Psi}{\partial\sigma^i}(\theta, 0) + F_i\left(\theta, \Psi(\theta, 0), \frac{\partial\Psi}{\partial\sigma}(\theta, 0), \ldots, \frac{\partial^{i-1}\Psi}{\partial\sigma^{i-1}}(\theta, 0)\right) \tag{20}$$

for some mappings F_i, $i \geqslant 2$.

In the implementation of our method, it will be convenient to relate the coefficients $\partial^i \Psi/\partial\sigma^i(\theta, 0)$ and $\partial^i \tilde{\Psi}/\partial\sigma^i(\theta, 0)$ obtained from nearby trajectories $r(\theta)$ and $\tilde{r}(\theta)$, respectively, of (15a). Suppose

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then that $r(\theta)$ and $\tilde{r}(\theta)$ are two solutions of (15a), and let $\epsilon(\theta) = \tilde{r}(\theta) - r(\theta)$. As before, let $\Psi(\theta,\sigma) = \psi(\theta, r(\theta)+\sigma)$ and $\tilde{\Psi}(\theta,\sigma) = \psi(\theta,\tilde{r}(\theta)+\sigma)$. Then

$$\frac{\partial\tilde{\Psi}}{\partial\sigma}(\theta,0) = \frac{\partial\psi}{\partial r}(\theta,\tilde{r}(\theta)) = \frac{\partial\psi}{\partial r}(\theta,r(\theta)+\epsilon(\theta)) = \sum_{j=0}^{\infty}\frac{\partial^{1+j}\psi}{\partial r^{1+j}}(\theta,r(\theta))\frac{\epsilon(\theta)^j}{j!},$$

where the last series converges for all $\theta \in [0,2\pi]$ provided $\epsilon(\theta)$ is sufficiently small. Using (18), we can write that

$$\frac{\partial\tilde{\Psi}}{\partial\sigma}(\theta,0) = \sum_{j=0}^{\infty}\frac{\partial^{1+j}\Psi}{\partial\sigma^{1+j}}(\theta,0)\frac{\epsilon(\theta)^j}{j!}.$$

Therefore, if $\partial\Psi/\partial\sigma(\theta,0),\ldots,\partial^N\Psi/\partial\sigma^N(\theta,0)$ are known, we can approximate $\partial\tilde{\Psi}/\partial\sigma(\theta,0)$ as

$$\frac{\partial\tilde{\Psi}}{\partial\sigma}(\theta,0) \approx \frac{\partial\Psi}{\partial\sigma}(\theta,0) + \frac{\partial^2\Psi}{\partial\sigma^2}(\theta,0)\epsilon(\theta) + \cdots + \frac{\partial^N\Psi}{\partial\sigma^N}(\theta,0)\frac{\epsilon(\theta)^N}{N!}$$

A similar argument shows that for $i \in \{1,2,\ldots\}$

$$\frac{\partial^i\tilde{\Psi}}{\partial\sigma^i}(\theta,0) = \sum_{j=0}^{\infty}\frac{\partial^{i+j}\Psi}{\partial\sigma^{i+j}}(\theta,0)\frac{\epsilon(\theta)^j}{j!}$$

and thus if $\partial^i\Psi/\partial\sigma^i(\theta,0),\ldots,\partial^N\Psi/\partial\sigma^N(\theta,0)$ are known we can approximate $\partial^i\tilde{\Psi}/\partial\sigma^i(\theta,0)$ as

$$\frac{\partial^i\tilde{\Psi}}{\partial\sigma^i}(\theta,0) \approx \sum_{j=0}^{N-i}\frac{\partial^{i+j}\Psi}{\partial\sigma^{i+j}}(\theta,0)\frac{\epsilon(\theta)^j}{j!} \tag{21}$$

provided $i \leqslant N$.

3.2. *Continuation algorithm*

With the previous constructions in mind, we are now ready to describe our continuation algorithm for computing the solution ψ to the center manifold (16).

(1) Let $N \geqslant 1$ be a fixed positive integer and let

$$\psi_0^N(\theta,r) = \sum_{i=1}^{N} e_i(\theta)\frac{r^i}{i!},$$

that is, ψ_0^N is the Nth-order Taylor approximation of ψ in r. To compute ψ_0^N, one can use the method in [5] to generate a Nth-order Taylor polynomial approximation of $\phi(w_1,w_2)$, say $\phi^N(w_1,w_2)$, and then $\psi_0^N(\theta,r) = \phi^N(r\cos\theta, r\sin\theta)$. For notational consistency, set $\Psi_0 = \psi$ and set $r_{-1}(\theta) = 0$ for $\theta \in \mathbb{R}$. The initial approximation ψ_0^N will be accepted in an annular region of the form

$$\{(\theta,r) : 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant r < r_0(\theta)\},$$

where $r_0(\theta)$ is the solution to (15a) with some prescribed initial condition $r_0(0) = \epsilon_0 > 0$. To compute accurate numerical solutions to r_0, we solve a BVP using (15a) with boundary conditions $r(0) = r(2\pi) = \epsilon_0$ and constant initial guess ϵ_0 on $[0,2\pi]$.

(2) Define $\Psi_1(\theta,\sigma) = \psi(\theta, r_0(\theta)+\sigma)$. From (17), Ψ_1 can be approximated by the truncated series

$$\Psi_1^N(\theta,\sigma) = \Psi_1(\theta,0) + \sum_{i=1}^{N}\frac{\partial^i\Psi_1}{\partial\sigma^i}(\theta,0)\frac{\sigma^i}{i!}$$

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for $|\sigma|$ small. To compute $\Psi_1(\theta,0) = \psi(\theta,r_0(\theta))$, we solve a BVP using (15b) with boundary conditions $z(0) = z(2\pi)$ and initial guess

$$\Psi_1(\theta,0) \approx \psi_0^N(\theta,r_0(\theta)).$$

Similarly, to compute the coefficient $\partial^i\Psi_1/\partial\sigma^i(\theta,0)$, $1 \leqslant i \leqslant N$, we solve a BVP using the ODE (20) with boundary conditions $\partial^i\Psi_1/\partial\sigma^i(0,0) = \partial^i\Psi_1/\partial\sigma^i(2\pi,0)$ and initial guess

$$\frac{\partial^i\Psi_1}{\partial\sigma^i}(\theta,0) \approx \frac{\partial^i\psi_0^N}{\partial r^i}(\theta,r_0(\theta)).$$

Having computed $\Psi_1(\theta,0), \partial\Psi_1/\partial\sigma(\theta,0),\ldots,\partial^N\Psi_1/\partial\sigma^N(\theta,0)$, we obtain an approximation $\psi_1(\theta,r)$ to $\psi(\theta,r)$ defined by

$$\psi_1(\theta,r) = \Psi_1^N(\theta,r-r_0(\theta)),$$

which is accepted in the region

$$\{(\theta,r) : 0 \leqslant \theta \leqslant 2\pi,\ r_0(\theta) \leqslant r < r_0(\theta)+\epsilon_1\} \tag{22}$$

for some desired $\epsilon_1 > 0$. In this way, we have extended our original approximation ψ_0 of ψ to the domain (22). Our running approximation of ψ is given by

$$\psi(\theta,r) \approx \begin{cases} \psi_0(\theta,r), & 0 \leqslant r < r_0(\theta), \\ \psi_1(\theta,r), & r_0(\theta) \leqslant r \leqslant r_0(\theta)+\epsilon_1 \end{cases}$$

for $\theta \in [0,2\pi]$.

(3) We now proceed to augment to our running approximation a mapping ψ_2 that will be defined on an annular region surrounding the domain of ψ_1 in the following way. We first compute the solution $r_1(\theta)$ to (15a) with initial condition $r_1(0) = r_0(0)+\epsilon_1$. This is performed by solving a BVP using (15a) with boundary conditions $r(0) = r(2\pi) = r_0(0)+\epsilon_1$ and taking the curve $r_0(\cdot)+\epsilon_1$ as an initial guess to r_1. Here, we note that to avoid overlapping domains of definition between ψ_1 and ψ_2, the domain (22) of ψ_1 is redefined to be

$$\{(\theta,r) : 0 \leqslant \theta \leqslant 2\pi, r_0(\theta) \leqslant r < r_1(\theta)\}.$$

(4) We now repeat Step 2 with $r_1(\theta)$ and build an approximation to $\Psi_2(\theta,\sigma) = \psi(\theta,r_1(\theta)+\sigma)$ of the form

$$\Psi_2^N(\theta,\sigma) = \Psi_2(\theta,0) + \sum_{i=1}^{N}\frac{\partial^i\Psi_2}{\partial\sigma^i}(\theta,0)\frac{\sigma^i}{i!},$$

for $0 \leqslant \sigma \leqslant \epsilon_2$ and $\epsilon_2 > 0$ small. To compute $\Psi_2(\theta,0) = \psi(\theta,r_1(\theta))$, we solve a BVP using (15b) with boundary conditions $z(0) = z(2\pi)$ and initial guess

$$\Psi_2(\theta,0) \approx \psi_1(\theta,r_1(\theta)).$$

Similarly, the coefficients $\partial^i\Psi_2/\partial\sigma^i(\theta,0)$, $1 \leqslant i \leqslant N$, are computed by solving a BVP using the ODE (20) with boundary conditions $\partial^i\Psi_2/\partial\sigma^i(0,0) = \partial^i\Psi_2/\partial\sigma^i(2\pi,0)$ and initial guess (see (21))

$$\frac{\partial^i\Psi_2}{\partial\sigma^i}(\theta,0) \approx \sum_{j=0}^{N-i}\frac{\partial^{i+j}\Psi_1}{\partial\sigma^{i+j}}(\theta,0)\frac{(r_1(\theta)-r_0(\theta))^j}{j!}.$$

Having computed $\Psi_2(\theta,0), \partial\Psi_2/\partial\sigma(\theta,0),\ldots,\partial^N\Psi_2/\partial\sigma^N(\theta,0)$, we obtain the approximation

$$\psi_2(\theta,r) = \Psi_2^N(\theta,r-r_1(\theta)),$$

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which is accepted in the region

$$\{(\theta, r) : 0 \leqslant \theta \leqslant 2\pi,\ r_1(\theta) \leqslant r < r_1(\theta) + \epsilon_2\}. \tag{23}$$

In this way, we extend our approximation of the true solution ψ to the annulus (23), and our running approximation is

$$\psi(\theta, r) \approx \begin{cases} \psi_0(\theta, r), & 0 \leqslant r < r_0(\theta), \\ \psi_1(\theta, r), & r_0(\theta) \leqslant r < r_1(\theta), \\ \psi_2(\theta, r), & r_1(\theta) \leqslant r \leqslant r_1(\theta) + \epsilon_2 \end{cases}$$

for $\theta \in [0, 2\pi]$.

(5) Steps 3–4 can now be iterated. Indeed, suppose we have computed an approximation $\psi_k(\theta, r)$ to $\psi(\theta, r)$ of the form $\psi_k(\theta, r) = \Psi_k^N(\theta, r - r_{k-1}(\theta))$ and defined in the region

$$\{(\theta, r) : 0 \leqslant \theta \leqslant 2\pi, r_{k-1}(\theta) \leqslant r < r_{k-1}(\theta) + \epsilon_k\}, \tag{24}$$

where $\Psi_k(\theta, \sigma) = \psi(\theta, r_{k-1}(\theta) + \sigma)$,

$$\Psi_k^N(\theta, \sigma) = \Psi_k(\theta, 0) + \sum_{i=1}^{N} \frac{\partial^i \Psi_k}{\partial \sigma^i}(\theta, 0)\frac{\sigma^i}{i!}$$

and $r_{k-1}(\theta)$ is the solution to (15a) with initial condition $r_{k-1}(0) = \Sigma_{i=0}^{k-1}\epsilon_i$. To extend our current approximation of ψ beyond the domain of ψ_k, we begin by computing the solution $r_k(\theta)$ to (15a) with initial condition $r_k(0) = r_{k-1}(0) + \epsilon_k$. This is performed by solving a BVP using (15a) with boundary conditions $r(0) = r(2\pi) = r_{k-1}(0) + \epsilon_k$ and initial guess $r_{k-1}(\cdot) + \epsilon_k$. Let now $\Psi_{k+1}(\theta, \sigma) = \psi(\theta, r_k(\theta) + \sigma)$ and form

$$\Psi_{k+1}^N(\theta, \sigma) = \Psi_{k+1}(\theta, 0) + \sum_{i=1}^{N} \frac{\partial^i \Psi_{k+1}}{\partial \sigma^i}(\theta, 0)\frac{\sigma^i}{i!}.$$

To compute $\Psi_{k+1}(\theta, 0)$, we solve a BVP using (15b) with boundary conditions $z(0) = z(2\pi)$ and initial guess

$$\Psi_{k+1}(\theta, 0) \approx \psi_k(\theta, r_k(\theta)).$$

Similarly, the coefficients $\partial^i \Psi_{k+1}/\partial\sigma^i(\theta, 0)$ are computed by solving a BVP using the ODE (20) with boundary conditions $\partial^i \Psi_{k+1}/\partial\sigma^i(0, 0) = \partial^i \Psi_{k+1}/\partial\sigma^i(2\pi, 0)$ and initial guess

$$\frac{\partial^i \Psi_{k+1}}{\partial \sigma^i}(\theta, 0) \approx \sum_{j=0}^{N-i} \frac{\partial^{i+j} \Psi_k}{\partial \sigma^{i+j}}(\theta, 0)\frac{(r_k(\theta) - r_{k-1}(\theta))^j}{j!}.$$

Having computed $\Psi_{k+1}(\theta, 0), \partial\Psi_{k+1}/\partial\sigma(\theta, 0), \ldots, \partial^N \Psi_{k+1}/\partial\sigma^N(\theta, 0)$, we obtain the approximation

$$\psi_{k+1}(\theta, r) = \Psi_{k+1}^N(\theta, r - r_k(\theta)),$$

which is accepted in the region

$$\{(\theta, r) : 0 \leqslant \theta \leqslant 2\pi,\ r_k(\theta) \leqslant r < r_k(\theta) + \epsilon_{k+1}\}. \tag{25}$$

To avoid overlapping the domains of ψ_k and ψ_{k+1}, the domain of definition (24) of ψ_k is redefined to be

$$\{(\theta, r) : 0 \leqslant \theta \leqslant 2\pi, r_{k-1}(\theta) \leqslant r \leqslant r_k(\theta)\}.$$

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After iterating Steps 3–4 a $k \geqslant 1$ number of times, we obtain the following piecewise smooth approximation to $\psi(\theta, r)$:

$$\psi(\theta, r) \approx \tilde{\psi}_k(\theta, r) := \begin{cases} \psi_0(\theta, r), & 0 \leqslant r < r_0(\theta), \\ \psi_1(\theta, r), & r_0 \leqslant r < r_1(\theta), \\ \vdots & \vdots \\ \psi_k(\theta, r), & r_{k-1} \leqslant r \leqslant r_k(\theta) \end{cases} \tag{26}$$

for $\theta \in [0, 2\pi]$. Let us make a few remarks about our algorithm.

Remark 3.1

(1) The coefficients $\partial^i \Psi / \partial \sigma^i(\theta, 0)$, $i = 0, 1, \ldots, N$, are computed by solving boundary value problems for mainly two reasons: (1) they are known *a priori* to be periodic, and (2) we have good approximations of them from the previously computed coefficients (21).

(2) The main computational effort of our method is in computing the coefficients $\partial^i \Psi / \partial \sigma^i(\theta, 0)$. From (20), we see that the ODE for $\partial^i \Psi / \partial \sigma^i(\theta, 0)$ is linear in $\partial^i \Psi / \partial \sigma^i(\theta, 0)$ and is polynomial in the previously computed $\Psi(\theta, 0), \ldots, \partial^{i-1} \Psi / \partial \sigma^{i-1}(\theta, 0)$. Hence, a way to speed up the computation is to solve for the coefficients $\partial^i \Psi / \partial \sigma^i(\theta, 0)$ order by order. This can result in computational savings when the right-hand side of (20) is complicated to evaluate or when $n \times N$ is large.

(3) When the w dynamics are given by the harmonic oscillator

$$\dot{w}_1 = -w_2$$
$$\dot{w}_2 = w_1,$$

the computation of the radial curves $r(\theta)$ is trivial and is given by the constant curves $r(\theta) = r(0)$. In this case, there is no need to redefine the outer boundary of the successive approximations ψ_i when going from one annulus to another.

(4) For the algorithm to produce a meaningful approximation to ψ, the domain on which the approximation (26) is defined, namely $\{(\theta, r) : 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant r \leqslant r_k(\theta)\}$, must be contained in the cylinder $\theta \in [0, 2\pi]$, $|r| \leqslant \epsilon$ on which ψ is defined. Because ϵ is not known *a priori*, the algorithm must proceed from r_{k-1} to r_k by taking small increments $r_k(0) - r_{k-1}(0) = \epsilon_k$ and choosing $r_0(0)$ sufficiently small.

To end this section, we prove that the sequence of approximations $\{\tilde{\psi}_k\}_{k=1}^{\infty}$ obtained from (26) converges uniformly to ψ.

Theorem 3.1

Suppose that ψ is defined on the cylinder $\Omega = \{(\theta, r) : 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant r \leqslant \epsilon\}$ and let $\tilde{\epsilon} < \epsilon$ be chosen so that if $r : [0, 2\pi] \to \mathbb{R}$ is a trajectory of (15a) with $r(0) \leqslant \tilde{\epsilon}$, then $r(\theta) < \epsilon$. Let $\tilde{\psi}_k$ be defined as in (26), with radial step-size $r_j(0) - r_{j-1}(0) = \epsilon_j := (1/(k+1))\tilde{\epsilon}$, for $j = 0, 1, \ldots, k$. Then $\tilde{\psi}_k \to \psi$ uniformly in $\tilde{\Omega} = \{(\theta, r) : 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant r \leqslant r_k(\theta)\}$.

Proof

We first note that the existence of $\tilde{\epsilon}$ follows by Lyapunov stability of the exosystem. By construction, $r_j(0) = ((j+1)/(k+1))\tilde{\epsilon}$ for $j = 0, 1, \ldots, k$, and in particular $r_k(0) = \tilde{\epsilon}$, thereby rendering the domain $\tilde{\Omega}$ independent of k. Also, we note that by shrinking $\tilde{\epsilon}$ if necessary, by Gronwall's lemma it follows that

$$|\tilde{r}(\theta) - \bar{r}(\theta)| \leqslant |\tilde{r}(0) - \bar{r}(0)| e^{K\theta}, \tag{27}$$

for all trajectories $\theta \mapsto \tilde{r}(\theta)$ and $\theta \mapsto \bar{r}(\theta)$ of (15a) such that $0 \leqslant \tilde{r}(0), \bar{r}(0) \leqslant \tilde{\epsilon}$, where K is a Lipschitz constant independent of θ.

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Now by definition, we have that $\Psi_j(\theta,\sigma) = \psi(\theta, r_{j-1}(\theta) + \sigma)$ and therefore

$$\frac{\partial^{N+1}\Psi_j}{\partial\sigma^{N+1}}(\theta,\sigma) = \frac{\partial^{N+1}\psi}{\partial r^{N+1}}(\theta, r_{j-1}(\theta)+\sigma).$$

Hence, by Taylor's theorem with integral remainder,

$$\begin{aligned}\Psi_j(\theta,\sigma) &= \Psi_j^N(\theta,\sigma) + \frac{1}{N!}\int_0^\sigma (\sigma-\tau)^N \frac{\partial^{N+1}\Psi_j}{\partial\sigma^{N+1}}(\theta,\tau)\,\mathrm{d}\tau \\ &= \Psi_j^N(\theta,\sigma) + \frac{1}{N!}\int_0^\sigma (\sigma-\tau)^N \frac{\partial^{N+1}\psi}{\partial r^{N+1}}(\theta, r_{j-1}(\theta)+\tau)\,\mathrm{d}\tau.\end{aligned}$$

Let $C = \max_{(\theta,r)\in\Omega} \|\partial^{N+1}\psi/\partial r^{N+1}(\theta,r)\|$, and we note that C exists by continuity of $\partial^{N+1}\psi/\partial r^{N+1}$ on the compact set Ω. We therefore have that

$$\|\Psi_j(\theta,\sigma) - \Psi_j^N(\theta,\sigma)\| \leqslant C\frac{|\sigma|^{N+1}}{(N+1)!}$$

provided $0 \leqslant \sigma \leqslant r_j(\theta) - r_{j-1}(\theta)$ for $\theta \in [0,2\pi]$, for all $j = 0,1,\ldots,k$. Now because $r_j(0) \leqslant \tilde{\epsilon}$ for $j = 0,1,\ldots,k$, it follows by (27) that

$$|r_j(\theta) - r_{j-1}(\theta)| \leqslant |r_j(0) - r_{j-1}(0)|e^{K\theta} \leqslant \frac{1}{k+1}\tilde{\epsilon}e^{2\pi K}.$$

Therefore, given $(\theta,r) \in \tilde{\Omega}$, say that $r_{j-1}(\theta) \leqslant r < r_j(\theta)$ for some $j \in \{0,1,\ldots,k\}$, it follows that

$$\begin{aligned}\|\psi(\theta,r) - \tilde{\psi}_k(\theta,r)\| &= \|\Psi_j(\theta, r - r_{j-1}(\theta)) - \psi_j(\theta, r - r_{j-1}(\theta))\| \\ &= \|\Psi_j(\theta, r - r_{j-1}(\theta)) - \Psi_j^N(\theta, r - r_{j-1}(\theta))\| \\ &\leqslant \frac{C}{(N+1)!}|r - r_{j-1}(\theta)|^{N+1} \\ &\leqslant \frac{C}{(N+1)!}\left(\frac{1}{k+1}\tilde{\epsilon}e^{2\pi K}\right)^{N+1}.\end{aligned}$$

Hence, $\|\psi(\theta,r) - \tilde{\psi}_k(\theta,r)\| \to 0$ as $k \to \infty$ uniformly in $\tilde{\Omega}$. This completes the proof. □

4. EXAMPLES

In this section, we present examples illustrating our method.

Example 4.1

In this example, we take a linear dynamical system of the form (12), whose center manifold is easily computed, and perform a nonlinear change of coordinates and obtain a nonlinear system on which we apply our method. The true solution to the nonlinear system is then readily available, and we can compare the approximations produced by our method with the true solution. Consider then the linear dynamical system

$$\begin{aligned}\dot{x}_1 &= x_2 + \frac{1}{2}w_1 + \frac{1}{2}w_2 \\ \dot{x}_2 &= x_3 + \frac{1}{3}w_1 + \frac{2}{3}w_2 \\ \dot{x}_3 &= -x_1 - \frac{1}{2}w_1 + \frac{1}{2}w_2 \\ \dot{w}_1 &= -w_2 \\ \dot{w}_2 &= w_1.\end{aligned}$$

The center manifold equation for this system in the unknown mapping $x = \phi(w)$ is

$$\frac{\partial \phi}{\partial w}(w) S w = C\phi(w) + Dw,$$

where $x = (x_1, x_2, x_3)$, $w = (w_1, w_2)$, and

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

It is straightforward to verify that $\phi(w) = (-1/3w_1, -1/2w_1 - 1/6w_2, -1/2w_1 - 1/6w_2)$ is the unique solution to the center manifold equation for this system. Consider the coordinate change $z = Z(x) = (-3x_1, 9x_1 - 6x_2, -x_2 + x_3 + \rho(-3x_1, 9x_1 - 6x_2))$, where $\rho : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. The system in (z, w) coordinates takes the form of (12) with matrix B having eigenvalues $-1, 1/2 \pm \sqrt{3}i$. By direct substitution, the solution to the center manifold equation of the system in (z, w) coordinates is $z = Z(\phi(w)) = (w_1, w_2, \rho(w_1, w_2))$, that is, the graph of the function ρ. For purposes of illustration, we take the egg carton shaped function $\rho(w_1, w_2) = \sin(w_1)\sin(w_2)$, whose graph is shown in Figure 1 (left). The patchy approximation $\tilde{\psi}_k(w_1, w_2) = (w_1, w_2, \tilde{\rho}(w_1, w_2))$ computed with our method with $k = 20$ annular regions of thickness $\epsilon = 0.25$ and of order $N = 2$ is shown in Figure 1 (center). The error between the patchy approximation and the true solution is shown in Figure 1 (right). □

Example 4.2

The following is an example of the rapidly growing behavior of polynomial approximations that can lead to destabilizing effects in an output tracking controller. We proceed as in Example 4.1 but instead use the volcano type function $\rho(w_1, w_2) = \sin(w_1^2 + w_2^2)e^{1-w_1^2-w_2^2}$, whose graph is shown in Figure 2 (left). The patchy approximation and the error in using the patchy approximation are shown in Figure 2 (center and right). The patchy approximation is constructed with $k = 60$ annular regions of thickness $\epsilon = 0.05$ and of order $N = 1$; that is, we only use a first-order Taylor series in the radial direction. For this example, polynomial approximations of orders up to 30 where tested on the domain in consideration, and it was verified that as one increases the order of the polynomial approximation the error in fact increases. □

Example 4.3

Consider the inverted pendulum cart system from Example 2.1 with two-dimensional exosystem given by

$$\dot{w} = s(w) = \begin{pmatrix} w_2 \\ -w_1 - aw_1^3 \end{pmatrix}, \qquad (28)$$

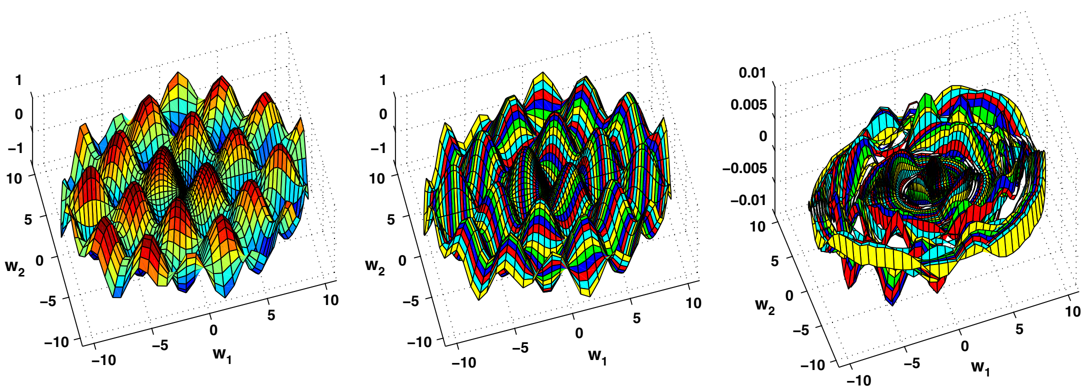

Figure 1. True solution $\rho(w_1, w_2) = \sin(w_1)\sin(w_2)$ (left); patchy solution $\tilde{\rho}(w_1, w_2)$ with degree $N = 2$, $k = 20$ annular regions, and radial step-size $\epsilon = 0.25$ (center); error $\rho(w_1, w_2) - \tilde{\rho}(w_1, w_2)$ (right).

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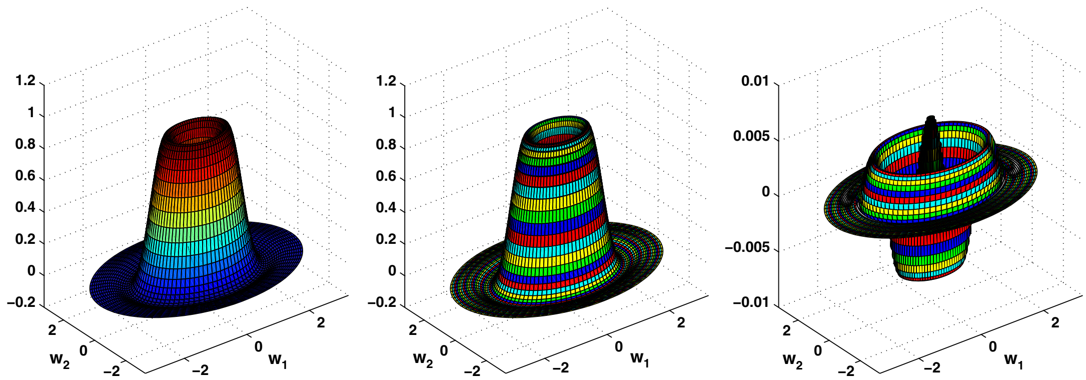

Figure 2. True solution $\rho(w_1, w_2) = \sin(w_1^2 + w_2^2)e^{1-w_1^2-w_2^2}$ (left); patchy solution $\tilde{\rho}(w_1, w_2)$ with degree $N = 1$, $k = 60$ annular regions, and radial step-size $\epsilon = 0.05$ (center); error $\rho(w_1, w_2) - \tilde{\rho}(w_1, w_2)$ (right).

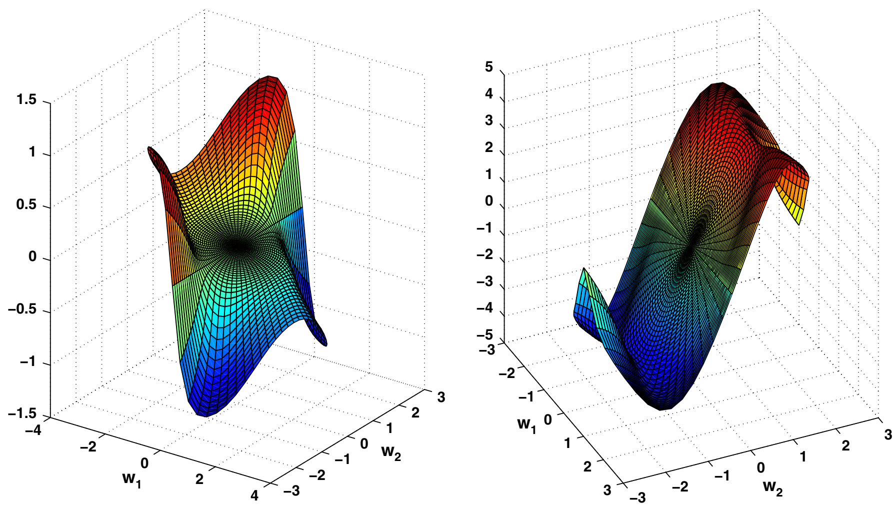

Figure 3. Patchy approximation to $\phi_1(w_1, w_2)$ (left) and $\phi_2(w_1, w_2)$ (right) with degree $N = 2$, $k = 40$ annular regions, and radial step-size $\epsilon = 0.05$.

where $a > 0$. System (28) is a special case of the unforced Duffing's oscillator with no damping [12]. The equilibrium $w = 0$ of (28) is a center.

Let $p(w) = -w_1$. As in Example 2.1, let (z, ξ) denote the standard normal coordinates, where $z = (x_3, x_4 + (x_2/\ell)\cos(x_3))$ and $\xi = (h(x), L_f h(x)) = (x_1, x_2)$. Following the notation at the beginning of Section 2, let $\varphi_1(w) = -p(w) = w_1$ and $\varphi_2(w) = -L_s p(w) = w_2$. Then the differential equation (8) for this system is given by

$$\begin{aligned}
\dot{z}_1 &= z_2 - \frac{1}{\ell} w_2 \cos(z_1) \\
\dot{z}_2 &= \frac{g}{\ell}\sin(z_1) - \frac{1}{\ell} z_2 w_2 \sin(z_1) + \frac{1}{\ell^2} w_2^2 \sin(z_1)\cos(z_1) \\
\dot{w}_1 &= w_2 \\
\dot{w}_2 &= -w_1 - a w_1^3.
\end{aligned} \tag{29}$$

A patchy approximation to the solution $\phi(w_1, w_2) = (\phi_1(w_1, w_2), \phi_2(w_1, w_2))$ of the center manifold equation of (29) was computed and is illustrated in Figure 3. The data $g = 10$, $\ell = 1/3$, and $a = 1/4$ were used. The solution was computed with $k = 40$ annular regions, and the radial

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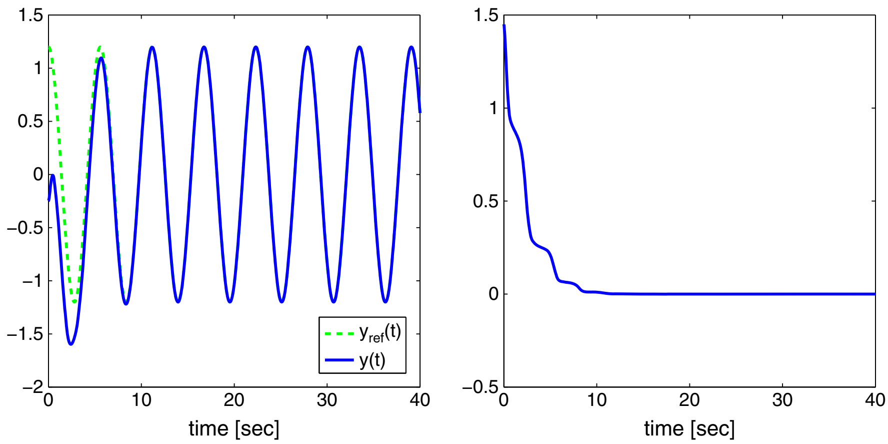

Figure 4. Output $y(t) = x_1(t)$ and reference $y_{\text{ref}}(t) = w_1(t)$ (left), and tracking error $e(t) = y(t) - y_{\text{ref}}(t)$ (right).

step at the initial angle $\theta = 0$ was taken to be $\epsilon_i = 0.05$, $i = 1, 2, \ldots, 40$. The order of the radial Taylor polynomials was chosen as $N = 2$.

The computed patchy approximation to the center manifold PDE for (29) is used in an output tracking controller of the form

$$\alpha(z, w) = \kappa(w) + K((z, \xi) - \pi(w)),$$

where $\pi(w) = (\phi(w), \varphi(w))$ and $\kappa(w) = u_e(\pi(w), w)$, and where the gain matrix K is chosen as the solution to a linear-quadratic regulator problem for the linearization of the inverted pendulum. In the linear-quadratic regulator problem, the matrices $Q = \text{diag}(4, 4, 4, 4)$ and $R = 1$ were chosen. A simulation is performed in which the pendulum is initialized at an angle of 15° from the vertical, and the cart is initialized at -0.25 from the origin. The reference trajectory, $y_{\text{ref}}(t) = w_1(t)$, is chosen with initial condition $y_{\text{ref}}(0) = 1.2$. The results of the simulation are shown in Figure 4. □

5. CONCLUSION

We have presented a method to compute solutions to the FBI equations of real analytic control-affine systems with two-dimensional exosystems. Our technique is based on the periodic nature of two-dimensional analytic center manifolds. In comparison with direct Taylor polynomial approximations [5, 6], our method lessens the computational effort needed to produce approximate solutions by taking into account the periodicity of the center manifold. We proved that our method generates a sequence of approximations converging uniformly to the true solution.

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