2016 American Control Conference (ACC)
Boston Marriott Copley Place
July 6-8, 2016. Boston, MA, USA

On almost equitable partitions and network controllability

Cesar O. Aguilar[1] and Bahman Gharesifard[2]

Abstract— **The main contribution of this paper are two new general necessary conditions for controllability of multi-agent networked control systems involving almost equitable partitions, along with an extension of a well-known symmetry condition to weighted digraphs and multi-input broadcast control signals. The new necessary conditions identify leader selections that break all "symmetries" induced by an almost equitable partition, and in particular genuine graph symmetries, but yet induce uncontrollable dynamics. The results are illustrated on non-trivial examples whose controllability properties are fully characterized by these conditions.**

I. Introduction

Equitable partitions have been used recently to provide necessary conditions for controllability of multi-agent networked control systems with multiple inputs [15], analyze the controllability of single-input multi-agent systems [10], obtain upper and lower bounds for the controllable subspace once the control nodes are selected [18], study model-reduction [13], and obtain sufficient conditions for the disturbance decoupling problem [12], [14]. Equitable partitions play an important role in studying spectral properties of the adjacency matrix, see for example [8, Section 9.3], and the more general notion of *almost* equitable partitions reveal spectral properties of the Laplacian matrix [3]. Equitable partitions appear also in the study of synchrony and pattern formation in coupled cell networks [16], [9], although in that setting they are referred to as "balanced" partitions.

In this paper, we consider weighted digraphs and the effect of almost equitable partitions on the controllability properties of the *out*-degree Laplacian matrix. Our main objective is to study the so-called leader-selection controllability problem and in characterizing the set of nodes from which a given networked control system is controllable/uncontrollable; we focus on the class of Laplacian leader-follower control systems [17], [15], [4], [2], although our results are applicable to alternative graph matrices (e.g., adjacency, normalized Laplacian, etc.). Our results allow us to go beyond the uncontrollable pairs that can be identified using symmetries of the graph, as in [17], [15], [4]. The key property of (almost) equitable partitions is that they induce invariant subspaces of the graph matrix, which therefore impose constraints on the leader-selection controllability problem for a networked multi-agent system. This work includes parts of our recent results in [1], extended here to the class of weighted digraphs with normal Laplacian matrix. The proofs are omitted for reasons of space and will appear elsewhere.

[1]Cesar O. Aguilar is with the Department of Mathematics, California State University, Bakersfield, CA 93311, USA caguilar24@csub.edu. The first author is supported by the National Science Foundation under grant ECCS-1509966.

[2]Bahman Gharesifard is with the Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada bahman@mast.queensu.ca

Statement of Contributions

The main results of this paper are new general necessary conditions for controllability of the leader-follower Laplacian dynamics using the information on the eigenvectors of the Laplacian matrix provided by almost equitable partitions. Specifically, our first result (Theorem 5.1) generalizes the relationship between graph symmetries and uncontrollability to the multi-input, multi-broadcast, and weighted digraph case [15]. More importantly, we provide two new necessary controllability conditions (Theorem 5.2 and 5.3) characterizing uncontrollable leader-selections that break the inherent "symmetry" of an almost equitable partition. These latter results show that the existence of almost equitable partitions can induce uncontrollability in complicated ways. As a by product of our results, we derive useful properties of almost equitable partitions of quotient graphs that are of independent interest. Examples are chosen to illustrate the results.

II. Preliminaries

Throughout this paper, the standard basis vectors in $\mathbb{R}^n$ are denoted by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. The orthogonal complement of a set $S \subset \mathbb{R}^n$ under the standard inner product on $\mathbb{R}^n$ will be denoted by $S^\perp$. The transpose of a real matrix $\mathbf{M}$ is denoted $\mathbf{M}^T$ and the conjugate transpose denoted by $\mathbf{M}^*$ if $\mathbf{M}$ has complex entries. The $n \times n$ identity matrix is denoted by $\mathbf{I}_n$. The column space of a matrix $\mathbf{M}$ will be denoted by $\operatorname{img}(\mathbf{M})$.

The set of binary vectors of length n will be denoted by $\{0,1\}^n$. Let $\mathbf{b} \in \{0,1\}^n$. We denote by $\chi(\mathbf{b})$ the characteristic indices of $\mathbf{b}$, i.e., $\chi(\mathbf{b}) \subseteq \{1, 2, \ldots, n\}$ is the set of indices where $\mathbf{b}$ is non-zero. Conversely, given any set $C \subseteq \{1, 2, \ldots, n\}$ the characteristic vector $\mathbf{c} \in \{0,1\}^n$ of C satisfies $\chi(\mathbf{c}) = C$. The all ones vector in $\{0,1\}^n$ is denoted by $\mathbf{1}_n$ and the all zeros vector is denoted by $\mathbf{0}_n$.

We recall some facts from linear algebra [6]. Let V be an inner product space and let $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be a diagonalizable linear operator. If W is a T-invariant subspace then the restriction of T to W, denoted by $\mathsf{T}|_\mathsf{W}$, is also diagonalizable. Hence, there exists a basis $\{v_1, v_2, \ldots, v_k\}$ for W that are eigenvectors of $\mathsf{T}|_\mathsf{W}$. The eigenvectors $v_1, v_2, \ldots, v_k$ are naturally eigenvectors of T, and thus there exists a basis for W consisting of eigenvectors of T. Now, since W is T-invariant, the orthogonal complement $\mathsf{W}^\perp$ is T^*-invariant,

978-1-4673-8681-4/$31.00 ©2016 AACC

where T^* denotes the adjoint of T. If T is self-adjoint then $\mathsf{W}^\perp$ is also T-invariant. Hence, in this case there also exists a basis for $\mathsf{W}^\perp$ consisting of eigenvectors of T. Hence, in this case, the eigenvectors of T split into those contained in W and those contained in $\mathsf{W}^\perp$.

We say that T is a **normal** linear operator if $\mathsf{T}\mathsf{T}^* = \mathsf{T}^*\mathsf{T}$. It is known that if T is normal then T and T^* have the same eigenvectors. We note that self-adjoint operators are special cases of normal operators.

III. Problem statement

To state the main problem under study in this paper, we need some notions from graph theory. Let G be a direct graph, or *digraph*. The vertex set of G is denoted by $V = V(G)$ and its edge set is denoted by $E = E(G) \subseteq V \times V$. For simplicity, we assume throughout that $V = \{1, 2, \ldots, n\}$. The weight of the edge $(i, j) \in E$ will be denoted by a_{ij}, and if $(i, j) \notin E$ we set $a_{ij} = 0$. We consider graphs that contain no self-loops, i.e., $a_{ii} = 0$, for all i. A *path* in G is an ordered sequence of vertices such that any ordered pair of vertices appearing consecutively is an edge of G. We say that G is *strongly connected* if there is a path between any pair of distinct vertices, and this is a standing assumption throughout this paper.

Let G_1 and G_2 be two graphs with the same vertex set V. We say that G_1 is the *complement* of G_2 on V if two vertices in G_1 are adjacent if and only if they are not in G_2. A graph G' with $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$ is called a *subgraph* of G. For a subset $S \subset V(G)$, the *subgraph induced* by S is the subgraph of G with vertex set S and whose edges consist of all edges in G with vertices in S.

The *out-neighbors* of $i \in V$ is the set $\mathcal{N}_{\text{out}}(i) := \{j \in V \mid (i, j) \in E\}$ and the *out-degree* of i is

$$d_{\text{out}}(i) = \sum_{j \in \mathcal{N}_{\text{out}}(i)} a_{ij} = \sum_{j \in V} a_{ij}.$$

The *out-adjacency matrix* of G is the $n \times n$ matrix $\mathbf{A}$ defined as $(\mathbf{A})_{ij} = a_{ij}$. The *out degree matrix* of G is the $n \times n$ diagonal matrix $\mathbf{D}$ defined by $(\mathbf{D})_{ii} = d_{\text{out}}(i)$. The *out-Laplacian matrix* of G is the $n \times n$ matrix defined by $\mathbf{L} = \mathbf{D} - \mathbf{A}$. Since G is strongly connected, it is well-known that $\lambda = 0$ is a simple eigenvalue of $\mathbf{L}$ affording the eigenvector $\mathbf{1}_n := (1, 1, \ldots, 1) \in \mathbb{R}^n$. Henceforth, by adjacency and Laplacian matrix we mean the out-adjacency and out-Laplacian matrix. The in-neighbors, in-degree, in-adjacency, and in-Laplacian matrix are defined similarly. A digraph is called *regular* if the in-degree and out-degree of each vertex are equal.

A permutation σ of the vertex set $V = \{1, 2, \ldots, n\}$ is an *automorphism* of G if $(i, j) \in E$ if and only if $(\sigma(i), \sigma(j)) \in E$. The group of automorphisms of G will be denoted by $\Gamma = \text{Aut}(G)$. We say that G is *asymmetric* if Γ contains only the identity permutation, and is called *symmetric* otherwise. If $\mathbf{P}_\sigma$ denotes the permutation matrix associated to σ then it is well-known that $\sigma \in \Gamma$ if and only if $\mathbf{A}\mathbf{P}_\sigma = \mathbf{P}_\sigma \mathbf{A}$ (equivalently, $\mathbf{L}\mathbf{P}_\sigma = \mathbf{P}_\sigma \mathbf{L}$ since σ must preserve degrees).

A. The Leader-Selection Controllability Problem

Let $\mathbf{L}$ be the Laplacian matrix of a weighted digraph G on n vertices. The leader-selection controllability problem for $\mathbf{L}$ with $m \geq 1$ inputs is to find a matrix $\mathbf{B} \in \{0, 1\}^{n \times m}$ such that the linear control system

$$\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1}$$

is controllable. It is well-known that the pair $(\mathbf{L}, \mathbf{B})$ is controllable if and only if the smallest $\mathbf{L}$-invariant subspace containing $\text{img}(\mathbf{B})$, denoted by $\langle \mathbf{L}; \mathbf{B} \rangle := \text{img}([\mathbf{B} \;\; \mathbf{L}\mathbf{B} \;\; \cdots \;\; \mathbf{L}^{n-1}\mathbf{B}])$, is all of $\mathbb{R}^n$. An equivalent characterization of controllability is the Popov-Belevitch-Hautus (PBH) test [5].

Theorem 3.1: The pair $(\mathbf{L}, \mathbf{B})$ is uncontrollable if and only if there exists an eigenvector $\mathbf{w} \in \mathbb{C}^n$ of $\mathbf{L}^T$ such that $\mathbf{w}^*\mathbf{B} = \mathbf{0}_{1 \times m}$.

In this paper, we are interested in obtaining graph-theoretic conditions on the choices of $\mathbf{B}$ that lead to controllable/uncontrollable pairs $(\mathbf{L}, \mathbf{B})$. When $\mathbf{B}$ is a column vector, we write $\mathbf{b}$ instead of $\mathbf{B}$, and this amounts to considering the case of a broadcasted control signal at the nodes specified by $\chi(\mathbf{b})$.

IV. Almost Equitable Partitions of Weighted Digraphs

Let G be a weighted digraph with adjacency matrix $\mathbf{A} = (a_{ij})$. Given a subset $C \subseteq V$, we denote for each $i \in V$ the out-degree of i relative to C by

$$d_{\text{out}}(i, C) = \sum_{j \in C} a_{ij}.$$

Let $\pi = \{C_1, C_2, \ldots, C_k\}$ be a partition of the vertex set V, that is, $\bigcup_{i=1}^k C_i = V$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. Following [3], we say that π is an *almost equitable partition* (AEP) of G if for all *distinct* ordered pairs of cells (C_r, C_s) it holds that $d_{\text{out}}(i, C_s)$ is independent of $i \in C_r$. In this case, we denote by $\alpha_{rs} := d_{\text{out}}(i, C_s)$ for any $i \in C_r$. We note that, in general, $\alpha_{rs} \neq \alpha_{sr}$. We define $\alpha_{rr} = 0$ for all $r = 1, 2, \ldots, k$. The *characteristic matrix* of π is the $n \times k$ matrix $\mathbf{P}(\pi)$ whose jth column is the characteristic vector of the cell C_j. Clearly, $\text{img}(\mathbf{P}(\pi))$ is a k-dimensional subspace consisting of vectors that are constant on the cells $C_1, \ldots, C_k$, that is, if $\mathbf{x} \in \text{img}(\mathbf{P}(\pi))$ then for each C_i the components of $\mathbf{x}$ on C_i are equal. When π is understood, we will write $\mathbf{P}$ instead of $\mathbf{P}(\pi)$.

Remark 4.1: **(Equitable partition):** An almost equitable partition is called an *equitable partition* if the subgraph induced by each cell is a regular graph [8]. It is a straightforward exercise to show that if σ is an automorphism of G, then the vertex partition induced by the orbits of σ is an equitable partition of G. •

Given an almost equitable partition π of G, we define the *out-neighbor quotient graph* of G over π, denoted by G/π, as the weighted digraph with vertices $V(G/\pi) = \{C_1, C_2, \dots, C_k\}$, edge set $E(G/\pi) = \{(C_r, C_s) \mid \alpha_{rs} \neq 0\}$, and the weight of $(C_r, C_s) \in E(G/\pi)$ is α_{rs}. We let $\mathbf{A}_\pi$ and $\mathbf{L}_\pi$ denote the out-adjacency and out-Laplacian matrix of G/π, respectively. In particular, for $r \neq s$, we have that $(\mathbf{L}_\pi)_{r,s} = -\alpha_{r,s}$, and

$$(\mathbf{L}_\pi)_{r,r} = d_{\text{out}}(r) = \sum_{C_s \in V(G/\pi)} \alpha_{r,s}.$$

Remark 4.2: **(The case of symmetric digraphs):** We can naturally associate to a simple graph G an edge symmetric digraph, i.e., a digraph where there is an edge between i and j if and only if there is an edge between j and i, whose weights are all unity. In this case, the definitions of G/π, $\mathbf{A}_\pi$, and $\mathbf{L}_\pi$ coincide with those in [3]. •

The following theorem describes one way in which $\mathbf{L}$-invariant subspaces arise with respect to an almost equitable partition. The proof of the following is similar to [3], [12] and thus we omit it.

Theorem 4.1: **(Characterization of AEP for weighted digraphs):** Let $G = (V, E)$ be a weighted digraph with out-Laplacian matrix $\mathbf{L}$, let $\pi = \{C_1, C_2, \dots, C_k\}$ be a partition of V, and let $\mathbf{P}$ be the characteristic matrix of π. Then π is an almost equitable partition of G if and only if $\operatorname{img}(\mathbf{P})$ is $\mathbf{L}$-invariant, that is, there exists a $k \times k$ matrix $\mathbf{Q}$ such that

$$\mathbf{LP} = \mathbf{PQ}.$$

In this case, $\mathbf{Q} = \mathbf{L}_\pi = (\mathbf{P}^T\mathbf{P})^{-1}\mathbf{P}^T\mathbf{LP}$.

Hence, when π is an almost equitable partition, $\mathbf{L}_\pi$ is the matrix representation of the restriction of $\mathbf{L}$ to the subspace $\mathsf{W} = \operatorname{img}(\mathbf{P}(\pi))$ in the basis obtained from the columns of $\mathbf{P}(\pi)$. Thus, if $\beta_1 = \{\mathbf{c}_1, \dots, \mathbf{c}_k\}$ consists of the columns of $\mathbf{P}(\pi)$ and we extend β_1 to a basis $\beta = \beta_1 \cup \beta_2$ of $\mathbb{R}^n$ then the matrix $\mathbf{L}$ in the basis β takes the form

$$[\mathbf{L}]_\beta = \begin{bmatrix} \mathbf{L}_\pi & \mathbf{L}_{12} \\ \mathbf{0} & \mathbf{L}_{22} \end{bmatrix}.$$

We note that, even if $\mathbf{L}$ is symmetric, $\mathbf{L}_\pi$ is not generally symmetric, since β_1 is not an orthonormal basis of W. However, $\mathbf{L}|_{\mathsf{W}}$ is indeed self-adjoint. In any case, every eigenvector of $\mathbf{L}|_{\mathsf{W}}$ is naturally an eigenvector of $\mathbf{L}$, which leads to the following result.

Proposition 4.1: **(Spectrum of the restricted Laplacian):** Let G, $\mathbf{L}$, $\mathbf{L}_\pi$, and $\mathbf{P}$ be as in Theorem 4.1. Then $(\mathbf{v}, \lambda)$ is an eigenvector-eigenvalue pair of $\mathbf{L}_\pi$ if and only if $(\mathbf{Pv}, \lambda)$ is an eigenvector-eigenvalue pair of $\mathbf{L}$.

The following useful result describes how AEPs of the quotient graph induce AEPs of the original graph.

Theorem 4.2: **(AEPs of the quotient graph):** Let G be a weighted digraph and suppose that $\pi = \{C_1, C_2, \dots, C_k\}$ is an AEP of G. Suppose that $\rho = \{S_1, S_2, \dots, S_\ell\}$ is an AEP of the quotient graph G/π. Define the partition $\pi/\rho = \{\overline{C}_1, \overline{C}_2, \dots, \overline{C}_\ell\}$ of $V(G)$ by asking that $\overline{C}_j = \bigcup_{C_i \in S_j} C_i$ for $j = 1, 2, \dots, \ell$. Then π/ρ is an almost equitable partition of G.

In Theorem 4.2, the partition π/ρ is a coarsening of the partition π. In fact, in terms of invariant subspaces, Theorem 4.2 follows from the general fact that for a linear operator $\mathsf{T} : \mathsf{V} \to \mathsf{V}$, if W_1 is T-invariant and $\mathsf{W}_2 \subset \mathsf{W}_1$ is $\mathsf{T}|_{\mathsf{W}_1}$-invariant, then W_2 is T-invariant, where we have naturally identified W_2 with a subspace of V. In Theorem 4.2, $\mathsf{W}_1 = \operatorname{img}(\mathbf{P}(\pi))$, $\mathsf{W}_2 = \operatorname{img}(\mathbf{P}(\rho))$, and $\mathsf{T} = \mathbf{L}$. We illustrate Theorem 4.2 with an example that will be used throughout the paper.

Example 4.1: Let G be the unweighted and undirected graph shown in Figure 1. The automorphism group $\Gamma = \operatorname{Aut}(G)$, as computed by NAUTY [11], is generated by the transpositions $\tau_1 = (2\ 3)$, $\tau_2 = (7\ 8)$, $\tau_3 = (1\ 6)$, and $\tau_4 = (4\ 5)$. The automorphism $\pi = \tau_1\tau_2\tau_3\tau_4$ induces an equitable partition of the vertex set $V(G)$ whose cells are $C_1 = \{2, 3\}$, $C_2 = \{7, 8\}$, $C_3 = \{1, 6\}$, $C_4 = \{4, 5\}$, $C_5 = \{9\}$, and $C_6 = \{10\}$. For convenience, we denote the induced partition by $\pi = \{C_1, C_2, C_3, C_4, C_5, C_6\}$. The quotient graph G/π is shown in Figure 2, where the weights shown on the edges correspond to the out-degree of the nodes. It can be verified that $\rho = \{\{3, 4\}, \{5, 6\}, \{1\}, \{2\}\}$ is an equitable partition of G/π induced by the automorphism $\rho = (3\ 4)(5\ 6)$ of G/π. The quotient graph $(G/\pi)/\rho$ is displayed in Figure 3. The partition π/ρ is therefore

$$\pi/\rho = \{\{2, 3\}, \{7, 8\}, \{9, 10\}, \{1, 6, 4, 5\}\},$$

and it can be verified that π/ρ is an almost equitable partition of G but not an equitable partition. Hence, although π and ρ are equitable partitions, π/ρ may in general be only almost equitable. •

V. Controllability and Almost Equitable Partitions

Suppose that π is an almost equitable partition of G. Then, since $\operatorname{img}(\mathbf{P})$ is $\mathbf{L}$-invariant, it follows that $\operatorname{img}(\mathbf{P})^\perp = \ker(\mathbf{P}^T)$ is $\mathbf{L}^T$-invariant. For simplicity, if we assume that G is undirected, then $\mathbf{L}$ is a symmetric matrix and hence $\ker(\mathbf{P}^T)$ is also $\mathbf{L}$-invariant. From the direct sum decomposition $\mathbb{R}^n = \operatorname{img}(\mathbf{P}) \oplus \ker(\mathbf{P}^T)$, we may therefore partition the eigenvectors of $\mathbf{L}$ into those that are constant on the cells of π, i.e., are contained in $\operatorname{img}(\mathbf{P})$, and those that sum to zero on the cells of π, i.e., are contained in $\ker(\mathbf{P}^T)$. For the leader-selection controllability problem, this implies that when the leader nodes are selected so that the resulting control input matrix has columns that are constant on the cells of π, these columns will be orthogonal to every eigenvector of $\mathbf{L}$ in $\ker(\mathbf{P}^T)$. We formalize our preceding discussion with the following theorem.

Theorem 5.1: **(A necessary condition for controllability: constant on the cells):** Let G be a weighted digraph on $n \geq 2$ vertices and suppose that $\pi = \{C_1, C_2, \dots, C_k\}$ is an almost equitable partition of G. Assume that $\mathbf{b}_1, \dots, \mathbf{b}_m \in$

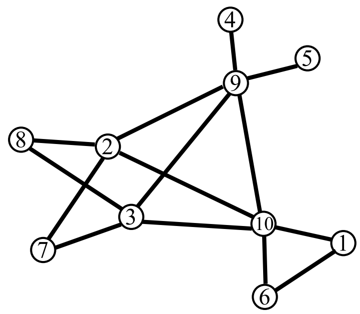

Fig. 1: G

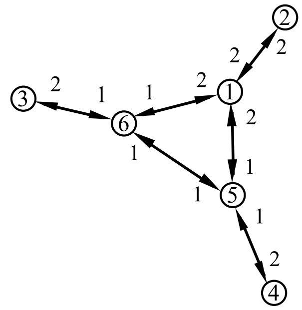

Fig. 2: G/π

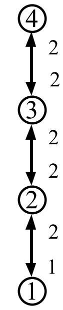

Fig. 3: $(G/\pi)/\rho$

$\{0,1\}^n$ are constant on the cells $C_1, C_2, \ldots, C_k$ and let $\mathbf{B} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_m\end{bmatrix}$. Then $\langle \mathbf{L}; \mathbf{B}\rangle \subseteq \operatorname{img}(\mathbf{P})$ and therefore $\dim\langle \mathbf{L}, \mathbf{B}\rangle \leq k$. In particular, if $k < n$ then $(\mathbf{L}, \mathbf{B})$ is uncontrollable.

Remark 5.1: **(Comparison with [18]):** The result in [18, Thm. 2] is a special case of Theorem 5.1, since there $\mathbf{B} = \begin{bmatrix}\mathbf{e}_{i_1} & \cdots & \mathbf{e}_{i_m}\end{bmatrix}$ and the first m cells of π are $C_j = \{i_j\}$, where $1 \leq j \leq m$, so that Theorem 5.1 is applicable. Moreover, if $\mathbf{L}$ is symmetric and π is an almost equitable partition, $\ker(\mathbf{P}^T)$ has a basis of (real) eigenvectors of $\mathbf{L}$, all of which are orthogonal to $\mathbf{B}$. •

Theorem 5.1 contains as a special case the situation of *leader-symmetric* selections, which we now define.

Definition 5.1: Let G be a weighted digraph. We say that $\mathbf{B} \in \{0,1\}^{n\times m}$ is *leader-symmetric* if there exists a non-trivial automorphism $\sigma \in \operatorname{Aut}(G)$ such that $\operatorname{img}(\mathbf{B}) \subseteq \operatorname{img}(\mathbf{P}(\sigma))$, that is, the columns of $\mathbf{B}$ are constant on the cells of σ. If $\mathbf{B}$ is not leader-symmetric we say that it is leader-asymmetric.

The following is an immediate corollary to Theorem 5.1.

Corollary 5.1: **(Leader symmetry):** Let G be a weighted digraph on $n \geq 2$ vertices. If $\mathbf{B} \in \{0,1\}^{n\times m}$ is leader-symmetric then $(\mathbf{L}, \mathbf{B})$ is an uncontrollable pair.

Remark 5.2: **(Comparison with [4]):** Proposition IV.8 in [4] is a special case of Corollary 5.1. Indeed, in [4], we have $\mathbf{B} = \begin{bmatrix}\mathbf{e}_{i_1} & \cdots & \mathbf{e}_{i_m}\end{bmatrix}$ and leader-symmetry amounts to the existence of a non-trivial automorphism σ such that $\sigma(i_j) = i_j$ for all $j = 1, \ldots, m$. In this case, the equitable partition induced by σ contains the cells $\{i_1\}, \{i_2\}, \ldots, \{i_m\}$, so that $\mathbf{B}$ is constant on the cells of σ. •

It is well-known that leader-symmetry is not necessary for uncontrollability [15], and thus it is natural to ask if the more general notion of AEPs, and in particular Theorem 5.1, is necessary for uncontrollability. In other words, we are interested in the case of $\operatorname{img}(\mathbf{B})$ not contained in $\operatorname{img}(\mathbf{P}(\pi))$ but $(\mathbf{L}, \mathbf{B})$ is uncontrollable. These binary vectors can therefore be thought of as "breaking the symmetry induced by π". To answer this question, we consider graphs on $n = 6$ vertices, since this is the minimal order for obtaining an asymmetric graph and graph symmetries induce equitable partitions. Of the eight asymmetric graphs on $n = 6$ vertices, the only four that have simple eigenvalues and contain uncontrollable leader selections are shown in Figure 4 (the other four are *essentially controllable*, see [2]). Graphs (a) and (b), and similarly graphs (c) and (d), are graph complements up to a relabeling of the vertices. Consequently, each pair has the same eigenvectors and thus the same controllability properties. Graphs (a) and (b) contain no AEPs, while graphs (c) and (d) contain only the AEPs $\pi_1 = \{\{5,6\},\{1,2,3,4\}\}$ and $\pi_2 = \{\{2,5\},\{1,3,4,6\}\}$, respectively. However, even for the graphs (c) and (d) with AEPs, there are uncontrollable binary vectors $\mathbf{b}$ that are not constant on the cells of the respective partition, i.e., they are not characterized by Theorem 5.1. To see how these uncontrollable binary vectors arise, consider the equitable partition $\pi_1 = \{\{5,6\},\{1,2,3,4\}\} =: \{C_1, C_2\}$ of the graph in Figure 4(c). By Theorem 5.1, the binary vector $\mathbf{b} = \mathbf{e}_5 + \mathbf{e}_6$, and its binary complement, yield uncontrollable dynamics. Let $\mathbf{w} \neq \mathbf{1}_n$ be an eigenvector of $\mathbf{L}$ contained in $\operatorname{img}(\mathbf{P}(\pi_1))$. Hence, $\mathbf{w} = \alpha_1\mathbf{c}_1 + \alpha_2\mathbf{c}_2$, where $\alpha_j \in \mathbb{R}$ and $\mathbf{c}_j$ is the characteristic vector of the cell C_j, for $j = 1, 2$. From $\mathbf{w}^T\mathbf{1}_n = 0$, it follows that $2\alpha_1 + 4\alpha_2 = 0$, or equivalently that $\alpha_1 + 2\alpha_2 = 0$. The *reduced* relation $\alpha_1 + 2\alpha_2 = 0$ induces new uncontrollable binary vectors not contained in $\operatorname{img}(\mathbf{P}(\pi_1))$ as follows. From the cell $C_1 = \{5,6\}$ we choose one non-zero component for $\mathbf{b}$ and from the cell $C_2 = \{1,2,3,4\}$ we choose two non-zero components for $\mathbf{b}$. By construction, $\mathbf{b}$ is not in $\operatorname{img}(\mathbf{P}(\pi_1))$, and clearly $\mathbf{w}^T\mathbf{b} = \alpha_1 + 2\alpha_2 = 0$, that is, $(\mathbf{L}, \mathbf{b})$ is uncontrollable. There are $\binom{2}{1}\binom{4}{2} = 12$ such choices for $\mathbf{b}$, and with the 2 uncontrollable binary vectors in $\operatorname{img}(\mathbf{P}(\pi_1))$ characterized by Theorem 5.1, this yields 14 non-trivial uncontrollable binary vectors. It can be verified that this completely characterizes all the uncontrollable binary vectors for the graph in Figure 4(c), and consequently for Figure 4(d) also. To formalize our previous discussion, we introduce the following definition.

Definition 5.2: Let G be a graph and let $\pi = \{C_1, C_2, \ldots, C_k\}$ be a partition of the vertices of G. We define

$$\gcd(\pi) = \gcd(|C_1|, |C_2|, \ldots, |C_k|)$$

and we say that π is *reducible* if $\gcd(\pi) \geq 2$.

With this definition we have the following theorem.

Theorem 5.2: **(A necessary condition for controllabil-**

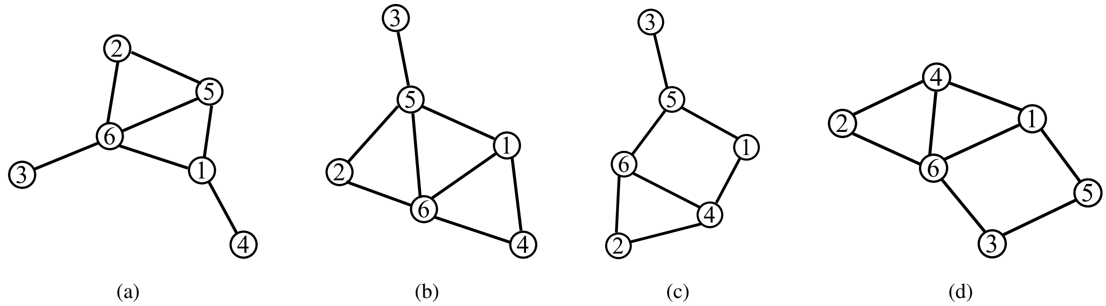

Fig. 4: (a), (b) Conditionally controllable graphs with no AEPs, (c), (d) Conditionally controllable graphs with AEPs

ity: symmetry-breaking leaders): Let G be a weighted digraph with normal Laplacian matrix $\mathbf{L}$. Let $\pi = \{C_1, C_2, \ldots, C_k\}$ be a reducible AEP of G, where $k \geq 2$, and let

$$q_j = \frac{|C_j|}{\gcd(\pi)}$$

for $j = 1, 2, \ldots, k$. Let $d \in \{1, \ldots, \gcd(\pi) - 1\}$ and let $\mathbf{b}$ be such that $|\chi(\mathbf{b}) \cap C_j| = dq_j$, for all $j = 1, 2, \ldots, k$. Then $(\mathbf{L}, \mathbf{b})$ is uncontrollable. In particular, there are

$$\sum_{d=1}^{\gcd(\pi)-1} \prod_{j=1}^{k} \binom{|C_j|}{dq_j}$$

such choices for $\mathbf{b}$.

We illustrate the use of Theorem 5.2 on the asymmetric 3-regular Frucht graph [7].

Example 5.1: Consider the asymmetric 3-regular Frucht graph on $n = 12$ vertices shown in Figure 5. We compute that the only non-trivial equitable partitions of the Frucht graph are

$$\pi_1 = \{\{3, 7, 10\}, \{1, 2, 4, 5, 6, 8, 9, 11, 12\}\}$$
$$\pi_2 = \{\{1, 5, 7, 12\}, \{3, 6, 9, 11\}, \{2, 4, 8, 10\}\}$$
$$\pi_3 = \{\{1, 5, 7, 12\}, \{2, 3, 4, 6, 8, 9, 10, 11\}\}.$$

We note that, since the Frucht graph is a regular graph, almost equitable partitions are automatically equitable partitions. In any case, consider the partition $\pi_1 = \{C_1, C_2\}$, with $\gcd(\pi_1) = 3$. There are two non-trivial binary vectors that are constant on the cells of π_1 and these are uncontrollable by Theorem 5.1. Let $\mathbf{w} \neq \mathbf{1}_n$ be an eigenvector of $\mathbf{L}$ contained in $\mathrm{img}(\mathbf{P}(\pi_1))$. Then $\mathbf{w} = \alpha_1 \mathbf{c}_1 + \alpha_2 \mathbf{c}_2$, where $\mathbf{c}_j$ is the characteristic vector of the cell C_j and $\alpha_j \in \mathbb{R}$, for $j = 1, 2$. From $\mathbf{w}^T \mathbf{1}_n = 0$, we have that $3\alpha_1 + 9\alpha_2 = 0$, or equivalently that $\alpha_1 + 3\alpha_2 = 0$. Hence, if $\mathbf{b}$ is such that $|\chi(\mathbf{b}) \cap C_1| = d$ and $|\chi(\mathbf{b}) \cap C_2| = 3d$, where $d \in \{1, 2\}$ then $\mathbf{w}^T \mathbf{b} = d\alpha_1 + 3d\alpha_2 = 0$. There are $\binom{3}{1}\binom{9}{3} + \binom{3}{2}\binom{9}{6} = 504$ such uncontrollable binary vectors. Hence, there are a total of 506 uncontrollable leader-selections corresponding to π_1.

Consider now the partition $\pi_2 = \{C_1, C_2, C_3\}$, with $\gcd(\pi_2) = 4$. There are $2^3 - 2 = 6$ non-trivial binary vectors in $\mathrm{img}(\mathbf{P}(\pi_2))$ and these are uncontrollable by Theorem 5.1. Let $\mathbf{w} \neq \mathbf{1}_n$ be an eigenvector of $\mathbf{L}$ contained

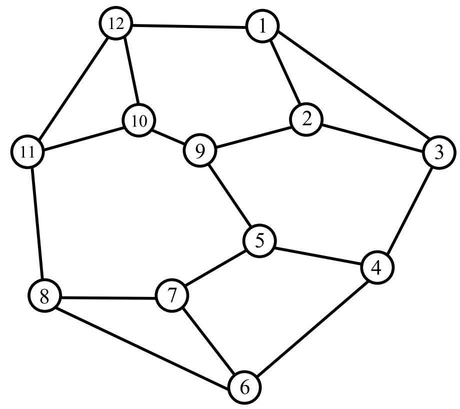

Fig. 5: Frucht graph

in $\mathrm{img}(\mathbf{P}(\pi_2))$. Then, $\mathbf{w} = \alpha_1 \mathbf{c}_1 + \alpha_2 \mathbf{c}_2 + \alpha_3 \mathbf{c}_3$, where $\mathbf{c}_j$ is the characteristic vector of C_j. Since $\mathbf{w}^T \mathbf{1}_n = 0$ we have that $4\alpha_1 + 4\alpha_2 + 4\alpha_3 = 0$, or equivalently $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Let $\mathbf{b}$ be such that $|\chi(\mathbf{b}) \cap C_j| = d$ for $j = 1, 2, 3$ where $d \in \{1, 2, 3\}$. Then $\mathbf{w}^T \mathbf{b} = d\alpha_1 + d\alpha_2 + d\alpha_3 = 0$. There are $\binom{4}{1}^3 + \binom{4}{2}^3 + \binom{4}{3}^3 = 344$ such choices for $\mathbf{b}$. Hence, there are 350 uncontrollable leader-selections corresponding to π_2.

Lastly, consider $\pi_3 = \{C_1, C_2\}$, with $\gcd(\pi_3) = 4$. Since π_2 is a finer partition than π_3, the uncontrollable binary vectors constant on the cells of π_3 have already been accounted for from π_2. Let $\mathbf{w} \neq \mathbf{1}_n$ be an eigenvector of $\mathbf{L}$ in $\mathrm{img}(\mathbf{P}(\pi_3))$. Then $\mathbf{w} = \alpha_1 \mathbf{c}_1 + \alpha_2 \mathbf{c}_2$ and thus $4\alpha_1 + 8\alpha_2 = 0$, or equivalently $\alpha_1 + 2\alpha_2 = 0$. Let $\mathbf{b}$ be such that $|\chi(\mathbf{b}) \cap C_1| = d$ and $|\chi(\mathbf{b}) \cap C_2| = 2d$ where $d \in \{1, 2, 3\}$. Then $\mathbf{w}^T \mathbf{b} = d\alpha_1 + 2d\alpha_2 = 0$. Not including the ones that have already been accounted for from π_2, for $d = 1$ there are $\binom{4}{1}\left[2\binom{4}{2}\right] = 48$ choices for $\mathbf{b}$, for $d = 2$ there are $\binom{4}{2}\left[2\binom{4}{0}\binom{4}{4} + 2\binom{4}{1}\binom{4}{3}\right] = 204$ choices for $\mathbf{b}$, and for $d = 3$ there are $\binom{4}{3}\left[2\binom{4}{4}\binom{4}{2}\right] = 48$ choices for $\mathbf{b}$. Hence, there are 300 uncontrollable leader-selections corresponding to π_3.

Hence, using Theorem 5.1 we can account for only 8 uncontrollable leader selections while with Theorem 5.2 we can account for 1148. However, we have computed that the Frucht graph has 1936 uncontrollable leader selections, and thus 780 remain unaccounted for. •

Remark 5.3: **(The case of no reducible AEP):** Let G be a graph on n vertices and let $\pi = \{C_1, C_2, \dots, C_k\}$ be a partition of G. Then $|C_1|+|C_2|+\cdots+|C_k| = n$ and therefore if n is prime then necessarily $\gcd(\pi) = 1$. Hence, in this case, G contains no reducible AEPs, and thus Theorem 5.2 is not applicable. •

Using the quotient graph, we give a further case in which the eigenvectors in $\mathrm{img}(\mathbf{P}(\pi))$ are orthogonal to binary vectors that are not orthogonal to eigenvectors in $\ker(\mathbf{P}^T(\pi))$. The situation arises when π, in contrast to Theorem 5.2, is not necessarily a reducible AEP.

Theorem 5.3: **(A necessary condition for controllability: symmetry-breaking leaders):** Let G be a weighted digraph with normal Laplacian matrix $\mathbf{L}$. Let $\pi = \{C_1, C_2, \dots, C_k\}$ be an almost equitable partition of G. Suppose that $\rho = \{S_1, S_2, \dots, S_\ell\}$ is a non-trivial almost equitable partition of the quotient graph G/π. Enumerate the elements of S_i as $S_i = \{C_{i,1}, C_{i,2}, \dots, C_{i,k_i}\}$ for $i = 1, 2, \dots, \ell$. For $i \in \{1, 2, \dots, \ell\}$ such that $k_i \geq 2$, suppose that $p_i = \gcd(|C_{i,1}|, \dots, |C_{i,k_i}|) \geq 2$ and let $q_{i,j} = \frac{|C_{i,j}|}{p_i}$, for $j = 1, 2, \dots, k_i$. For i such that $k_i \geq 2$, let $d_i \in \{1, 2, \dots, p_i - 1\}$ and suppose that $\mathbf{b}$ is such that $|\chi(\mathbf{b}) \cap C_{i,j}| = d_i q_{i,j}$. Then $(\mathbf{L}, \mathbf{b})$ is uncontrollable.

The next example shows how Theorem 5.3 can be used to account for uncontrollable leader selections not covered by Theorem 5.2.

Example 5.2: Consider again the asymmetric 3-regular Frucht graph on $n = 12$ vertices shown in Figure 5 and studied in Example 5.1. The Laplacian matrix of the quotient graph G/π_2 is

$$\mathbf{L}_{\pi_2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

and it is easy to see that $\rho = \{\{C_1\}, \{C_2, C_3\}\}$ is an equitable partition of $\mathbf{L}_{\pi_2}$. The partition π_2/ρ is precisely π_3 from Example 5.1. Now, from Theorem 5.3, if $\mathbf{b}$ is such that $|\chi(\mathbf{b}) \cap C_2| = 2$, $|\chi(\mathbf{b}) \cap C_3| = 2$, and $\chi(\mathbf{b}) \cap C_1$ is arbitrary, then $(\mathbf{L}, \mathbf{b})$ is uncontrollable. Such choices of $\mathbf{b}$ are not all characterized by Theorem 5.2. Indeed, applying Theorem 5.2 to π_2 directly as in Example 5.1, we must have that $|\chi(\mathbf{b}) \cap C_j| = d$ for $j = 1, 2, 3$ where $d \in \{1, 2, 3\}$. Hence, the case $|\chi(\mathbf{b}) \cap C_1| = 2$ is the only one that has been accounted for by Theorem 5.2, but the cases $|\chi(\mathbf{b}) \cap C_1| \in \{0, 1, 3, 4\}$ induce new uncontrollable leader selections characterized by Theorem 5.3. There are a total of $\binom{4}{2}\binom{4}{2}\left[\binom{4}{0} + \binom{4}{1} + \binom{4}{3} + \binom{4}{4}\right] = 360$ such choices for $\mathbf{b}$. •

VI. Conclusions

In this paper, we considered the leader-selection controllability problem for graphs. The main results of the paper are two new necessary conditions for controllability involving almost equitable partitions (Theorem 5.2 and Theorem 5.3). We also generalized the known results on the role of graph symmetries and uncontrollability to weighted digraphs and multiple-leaders (Theorem 5.1). The results were illustrated on non-trivial examples.

References

[1] C. Aguilar and B. Gharesifard. Almost equitable partitions and new necessary conditions for network controllability. *submitted*, 2015. available: http://www.csub.edu/∼caguilar24/papers/aep2015.pdf.

[2] C.O. Aguilar and B. Gharesifard. Graph controllability classes for the Laplacian leader-follower dynamics. *IEEE Transactions on Automatic Control*, 60(6):1611–1623, 2015.

[3] D.M. Cardoso, C. Delorme, and P. Rama. Laplacian eigenvectors and eigenvalues and almost equitable partitions. *European Journal of Combinatorics*, 28(3):665–673, 2007.

[4] A. Chapman, M. Nabi-Abdolyousefi, and M. Mesbahi. Controllability and observability of network-of-networks via Cartesian products. *IEEE Transactions on Automatic Control*, 59(10):2668–2679, Oct 2014.

[5] G. Dullerud and F. Paganini. *Course in Robust Control Theory*. Springer-Verlag, New York, 2000.

[6] S. Friedberg, A. Insel, and L. Spence. *Linear Algebra*. Prentice Hall, New Jersey, 4 edition, 2003.

[7] R. Frucht. Herstellung von graphen mit vorgegebener abstrakter gruppe. *Compositio Mathematica*, 6:239–250, 1939.

[8] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer, 2001.

[9] M. Golubitsky, I. Stewart, and A. Török. Patterns of synchrony in coupled cell networks with multiple arrows. *SIAM Journal on Applied Dynamical Systems*, 4(1):78–100, 2005.

[10] S. Martini, M. Egerstedt, and A. Bicchi. Controllability analysis of multi-agent systems using relaxed equitable partitions. *International Journal of Systems, Control and Communications*, 2(1):100–121, 2010.

[11] B. McKay and A. Piperno. Practical graph isomorphism, II. *Journal of Symbolic Computation*, 60:94–112, 2013.

[12] N. Monshizadeh. *Model reduction and control of complex systems*. PhD thesis, University of Groningen, 2013.

[13] N. Monshizadeh, H.L. Trentelman, and M.K. Camlibel. Projection-based model reduction of multi-agent systems using graph partitions. *Control of Network Systems, IEEE Transactions on*, 1(2):145–154, June 2014.

[14] N. Monshizadeh, S. Zhang, and M.K. Camlibel. Disturbance decoupling problem for multi-agent systems: A graph topological approach. *Systems & Control Letters*, 76:35–41, 2015.

[15] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt. Controllability of multi-agent systems from a graph-theoretic perspective. *SIAM Journal on Control and Optimization*, 48(1):162–186, 2009.

[16] I. Stewart, M. Golubitsky, and M. Pivato. Symmetry groupoids and patterns of synchrony in coupled cell networks. *SIAM Journal on Applied Dynamical Systems*, 2(4):609–646, 2003.

[17] H.G. Tanner. On the controllability of nearest neighbor interconnections. In *IEEE Conf. on Decision and Control*, pages 2467–2472, 2004.

[18] S. Zhang, M. Cao, and M.K. Camlibel. Upper and lower bounds for controllable subspaces of networks of diffusively coupled agents. *IEEE Transactions on Automatic Control*, 59(3):745–750, March 2014.