Sequences
Real Analysis
Review Questions

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Disclaimer: This is a list of questions to guide you through your studies. Not everything that is asked in these questions will actually be tested (due to time constraints), and conversely, there might be a question that is tested that was not explicitly covered by these questions. Use these questions only as a supplement to the questions in the lecture notes/homework and don't feel like you need to answer every question here to be 100% ready for your test. **Solutions to these questions will not be provided.**

1. The sequence (x_n) converges to L if ______________________________.
2. A sequence (x_n) is bounded if ______________________________.
3. Prove by definition that $\lim_{n\to\infty} \frac{n+1}{n^2} = 0$.
4. Prove by definition that $\lim_{n\to\infty} \frac{2n^3+n}{n^3+4} = 2$.
5. Prove by definition that $\lim_{n\to\infty} \frac{\cos(n)}{n^2-1} = 0$.
6. Let (x_n) be a sequence and let $L \in \mathbb{R}$. Suppose that
$$|x_n - L| \leq \frac{1}{n^2} \qquad \text{for all } n \geq 33$$
Prove by definition that (x_n) converges to L.
7. Prove that if (x_n) is convergent then it is bounded.
8. Suppose that $(x_n) \to L$ and $(y_n) \to M$. Define the sequence $z_n = 2x_n + 4y_n$. Prove **by definition** that (z_n) converges to $2L + 4M$.
9. Suppose that $(x_n) \to 0$ and suppose that (y_n) is bounded. Prove that $\lim\limits_{n\to\infty} x_n y_n = 0$.
10. Suppose that $(x_n) \to L$. Let (y_n) be a sequence such that for every $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that $|x_n - y_n| < \varepsilon$ for all $n \geq K$. Prove that (y_n) also converges to L.
11. Give examples of the following:
(a) Divergent sequences (x_n) and (y_n) such that $z_n = x_n y_n$ converges.
(b) Divergent sequences (x_n) and (y_n) such that $z_n = x_n y_n$ diverges.
(c) A divergent sequence (x_n) and a convergent sequence (y_n) such that $z_n = x_n y_n$ converges.
(d) A divergent sequence (x_n) and a convergent sequence (y_n) such that $z_n = x_n y_n$ diverges.
12. Use the Limit Theorems to prove that if (x_n) converges and $(x_n + y_n)$ converges then (y_n) converges. Give an example of two sequences (x_n) and (y_n) such that **both** (x_n) and (y_n) diverge but $(x_n + y_n)$ converges.
13. State the Monotone Convergence Theorem (MCT).
14. True or false, a convergent sequence is necessarily monotone? If it is true then prove it, or if it is false then give an example of a convergent sequence that is not monotone.
15. Give an example of a monotone sequence that diverges.
16. Give an example of a bounded sequence that diverges.
17. Consider the sequence (x_n) defined as $x_1 = 1$ and $x_{n+1} = \dfrac{n}{8n+1} x_n^4$ for $n \geq 1$.

(a) Prove that $0 \leq x_n \leq 1$ for all $n \geq 1$.

(b) Prove that (x_n) is monotone.

(c) What theorem allows you to conclude that (x_n) is convergent?

(d) If $L = \lim_{n\to\infty} x_n$, is it possible for $L > 1$? Explain.

(e) Find $\lim_{n\to\infty} x_n$.

18. We proved that **if** (x_n) converges to L **then** $(|x_n|)$ converges to $|L|$. The converse, however, is not true. Find an example of a sequence (x_n) such that $(|x_n|)$ converges but (x_n) is divergent.

19. Prove that if (x_n) converges to L then every subsequence of (x_n) converges to L.

20. Suppose that (x_n) is an increasing sequence and let (x_{n_k}) be a subsequence of (x_n).

    (a) Prove that if (x_{n_k}) is bounded above then (x_n) is also bounded above.

    (b) Are (x_{n_k}) and (x_n) convergent? Why?

    (c) Is it necessarily true that $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n$? Why?

21. Suppose that (x_n) is a decreasing sequence and let (x_{n_k}) be a subsequence of (x_n).

    (a) Prove that if (x_{n_k}) is bounded below then (x_n) is also bounded below.

    (b) Are (x_{n_k}) and (x_n) convergent? Why?

    (c) Is it necessarily true that $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n$? Why?

22. State the Bolzano-Weierstrass Theorem.

23. Show directly that if (x_n) is a Cauchy sequence then it is bounded.

24. Show that if (x_n) is convergent then it is a Cauchy sequence.

25. Suppose that (x_n) and (y_n) are sequences such that $|x_m - x_n| \leq |y_m - y_n|$ for all $n, m \in \mathbb{N}$. Show that if the sequence (y_n) is convergent then so is the sequence (x_n).

26. Let $0 < r < 1$. Show that if the sequence (x_n) satisfies $|x_n - x_{n+1}| < r^n$ for all $n \in \mathbb{N}$ then (x_n) is a Cauchy sequence and therefore convergent.