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Solutions to these questions will not be provided.

1. Series

1. A series $\sum x_n$ converges if ____.
2. Prove that if $\sum x_n$ converges then $\lim_{n\to\infty} x_n = 0$.
3. Does the series $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ converge?
4. Let (x_n) be a sequence of non-negative numbers. Prove that the series $\sum x_n$ converges if and only if the sequence (s_k) of partial sums is bounded. In this case, what is the value of $\sum x_n$?
5. Prove that the Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.
6. Prove that if $|r| < 1$ then the Geometric series $\sum_{n=0}^{\infty} r^n$ converges. What is the value of $\sum_{n=0}^{\infty} r^n$?
7. Let (x_n) and (y_n) be non-negative sequences and suppose that $x_n \leq y_n$ for all $n \in \mathbb{N}$. Prove that if $\sum y_n$ converges then $\sum x_n$ converges.
8. Is the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ convergent?
9. Suppose that $0 < x_n \leq 1$ for all $n \in \mathbb{N}$ and suppose that $\sum x_n$ converges. Is it necessarily true that $\sum x_n^2$ converges? Either prove that it is true or give a counterexample.
10. Suppose that $0 < x_n \leq 1$ for all $n \in \mathbb{N}$ and suppose that $\sum x_n$ converges. Is it necessarily true that $\sum \sqrt{x_n}$ converges? Either prove that it is true or give a counterexample.

2. Limits

1. Let $A \subset \mathbb{R}$. The point $c \in \mathbb{R}$ is a cluster point of A if ________________________________.
2. State the definition of the limit of a function $f : A \to \mathbb{R}$ at $c \in \mathbb{R}$.
3. Prove by definition that $\lim_{x\to 6} \frac{x^2-3x}{x+3} = 2$.
4. Prove by definition that $\lim_{x\to 1} \frac{x+1}{x+3} = \frac{1}{2}$.
5. State the sequential criterion for the limit of a function $f : A \to \mathbb{R}$ at c.
6. Give an example of a set $A \subset \mathbb{R}$, a function $f : A \to \mathbb{R}$, and a point $c \in \mathbb{R}$ where c is a cluster point of A, $\lim_{x\to c} f(x)$ exists but f is not well-defined at c.
7. Give an example of a set $A \subset \mathbb{R}$, a function $f : A \to \mathbb{R}$, and a point $c \in \mathbb{R}$ where f is bounded locally at c but $\lim_{x\to c} f(x)$ does not exist. For your example, prove that $\lim_{x\to c} f(x)$ does not exist.
8. Let $f : \mathbb{R} \to \mathbb{R}$ be the function
$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$
Prove that f has no limit at $c = 0$.
9. Let c be a cluster point of A. The function $f : A \to \mathbb{R}$ is bounded locally at c if ____________________.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is bounded locally at c and suppose that $g : \mathbb{R} \to \mathbb{R}$ converges to $L = 0$ at c. Prove that $\lim_{x \to c} f(x)g(x) = 0$.
11. Let c be a cluster point of $A \subset \mathbb{R}$. Suppose that $f, g : A \to \mathbb{R}$ have a limit at c and let $\alpha, \beta \in \mathbb{R}$. Prove that $h(x) = \alpha f(x) + \beta g(x)$ has a limit at c.
12. Let c be a cluster point of A. Suppose that $f : A \to \mathbb{R}$ satisfies $M_1 \le f(x) \le M_2$ for all $x \in A$, for some constants M_1 and M_2. Prove that if $\lim_{x \to c} f(x) = L$ then $M_1 \le L \le M_2$.
13. Let c be a cluster point of $A \subset \mathbb{R}$ and suppose that $f : A \to \mathbb{R}$ has limit L at c. Prove that if $L < 0$ then there exists a neighborhood $B_\delta(c) = (c - \delta, c + \delta)$ of c such that $f(x) < 0$ for all $x \in B_\delta(c) \cap A$.
14. State the Squeeze Theorem for functions and then prove it.
15. Prove that $\lim_{x \to 0} \cos(1/x)$ does not exist but $\lim_{x \to 0} x \cos(1/x) = 0$.

3. Continuity

1. The function $f : A \to \mathbb{R}$ is continuous at $c \in A$ if ____.
2. State the sequential criterion for a function $f : A \to \mathbb{R}$ to be continuous at $c \in A$.
3. Consider the continuous function $f(x) = x^2$ on $A = [0, 5]$. Fix $\varepsilon = 1$. Find the largest $\delta_1 > 0$ so that if $x \in (2 - \delta_1, 2 + \delta_1)$ then $|f(x) - f(2)| < \varepsilon$.
4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$$

   Prove that f is discontinuous at every point $c \in \mathbb{R}$.
5. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Prove that if $f(q) = g(q)$ for $q \notin \mathbb{Q}$ then $f(x) = g(x)$ for all $x \in \mathbb{R}$.
6. Let $A = \mathbb{R} \backslash \{0\}$ and consider the function $f : A \to \mathbb{R}$ defined by $f(x) = x^2 \sin(1/x)$.
   (a) Find $\lim_{x \to 0} f(x)$.
   (b) Is f continuous at $x = 0$? Why?
   (c) Extend the function f to a continuous function $F : \mathbb{R} \to \mathbb{R}$ so that $F(x) = f(x)$ for $x \neq 0$.
7. Given an example of a function $f : [a, b] \to \mathbb{R}$ that does not achieve a minimum nor a maximum in $[a, b]$.
8. State the Intermediate Value Theorem.
9. Prove that $f(x) = \cos(x)$ and $g(x) = x^2$ intersect at some point inside the interval $[0, \pi/2]$.
10. True or False: If $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded on $[a, b]$.
11. True or False: If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous then f is bounded.
12. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and assume that $f(a) < 0$ and $f(b) > 0$. Let $W = \{x \in [a, b] : f(x) < 0\}$ and let $w = \sup(W)$. Prove that $f(w) = 0$.
13. Prove using the Intermediate Value Theorem that you can slice a cheese pizza in half so that both halves have an equal amount of cheese.
14. The function $f : A \to \mathbb{R}$ is uniformly continuous on A if ____________________________________.

15. The function $f : A \to \mathbb{R}$ is a Lipschitz function on A if ____.

16. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be Lipschitz functions. Prove that $f + g$ is also Lipschitz on A.

17. Let $f : A \to \mathbb{R}$ be a Lipschitz function and suppose that $f(A) \subset B$. Prove that if $g : B \to \mathbb{R}$ is Lipschitz then the composite function $(g \circ f) : A \to \mathbb{R}$ is Lipschitz.

18. Prove that if $f : A \to \mathbb{R}$ is a Lipschitz function then f is uniformly continuous.

19. Give an example of a continuous function $f : A \to \mathbb{R}$ that is not uniformly continuous. Your example needs to explicitly state the domain A. For your example, prove that f is not uniformly continuous.

20. Give an example of a uniformly continuous function $f : A \to \mathbb{R}$ that is not Lipschitz. Your example needs to explicitly state the domain A. For your example, prove that f is not Lipschitz.