(A growing list of) Tips for writing proper mathematics

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Below is a list of tips to help you improve your mathematical writing. Learning how to write proper mathematics will help you become a better writer in general.

1. You should write solutions to homework and test problems under the assumption that the reader cannot guess what you meant to write but can only read and comprehend what you did write. Use **complete** English sentences, making clearly identifiable statements that have a clear meaning and that can be understood by anyone that is trained in reading mathematics.

2. **Do not display a sequence of equations without any explanation to what you are doing**. Here is a first example:

   Example 1 (Bad): *"We want to solve the equation, so*

$$\begin{aligned} x^2 + 5x - 2 &= 0 \\ x &= \frac{-5 \pm \sqrt{5^2 - 4(-2)}}{2} \\ x &= \frac{-5 \pm \sqrt{25 + 8}}{2}. \end{aligned}$$

   Here is a second example:
   Example 2 (Bad): *"We want to show that $f(n) = \frac{1}{n}$ is an injection so*

$$\begin{aligned} f(n) &= f(m) \\ \frac{1}{n} &= \frac{1}{m} \\ \frac{nm}{n} &= \frac{nm}{m} \\ n &= m \end{aligned}$$

   Instead, you need to **explain** what you are doing and what the key steps are. This means that you will need to write **words** and combine the words to produce a **sentence**. *In other words, you need to incorporate the mathematics into properly written sentences.* For example, we could write the first example as:

   Example 1 (Good): *"Using the quadratic formula to solve for x from the equation $x^2 + 5x - 2 = 0$ one obtains $x = \frac{-5 \pm \sqrt{33}}{2}$."*

   Notice that the equation and the solution for x were written as part of a sentence which ended with a period. The second example could be written as:

Example 2 (Good): *"To prove that $f(n) = \frac{1}{n}$ is an injection we must show that if $f(n) = f(m)$ then $n = m$. Hence, suppose that $f(n) = f(m)$. Then $\frac{1}{n} = \frac{1}{m}$ and after rearranging we obtain $n = m$."*

3. **Do not start a sentence with a math symbol**. For example: "f is a surjection if ...". Instead, you could use "**The function** f is a surjection if ..."

4. **Avoid the usage of the phrase "we know that"**. Usually, you know something because you are **assuming** it or because **it follows** from a theorem or basic identity, etc. Therefore, instead of saying "we know that" say **why** you know what you know. For example, if it is being assumed that f is an injection, instead of writing "We know that f is an injection and thus ..." you could say "**By assumption**, f is an injection and thus ...", or "**Since** f is an injection then ...". In many cases you do not even need to say "we know that". For example, "and we know that $(3, 2) \in \mathbb{N} \times \mathbb{N}$" could simply be written as "and $(3, 2) \in \mathbb{N} \times \mathbb{N}$".

5. **Do not use a math symbol to replace words in a sentence**. For example, "Therefore, $\exists$ an x such that ...", or "Then the $\mathcal{P}(S) = \{\emptyset, \{x\}, \dots, \}$". Instead, you should write "Therefore, **there exists a real number** x such that ...", and "Then the **power set of** S **is** $\mathcal{P}(S) = \dots$". Symbols such as $\exists$ and $\forall$ can be used in long displayed formulas, for example,

$$S = \{y \in \mathbb{R} \mid \exists\, x \in A \text{ such that } y = f(x)\}$$

or

$$S = \left\{y \in \mathbb{R} \mid y < \frac{n^2 + 1}{4n^3 - 2n},\ \forall\, n \in \mathbb{N}\right\}.$$

6. In many cases, it is better to use the word "**Let**" instead of "suppose" or "assume". For example, "Suppose we have a function $f : Q \to P$ defined by ...". Instead, use "Let $f : Q \to P$ be defined by ..." or "Consider the function $f : Q \to P$ defined by ...". As another example, instead of writing "Suppose that there exists a complex number $a + bi$ where $a, b \in \mathbb{R}$", write instead "**Let** $a + bi$ be a complex number.".

7. **Informal language/the unprofessional use of the English language in an academic setting:** Avoid the use **"so"** and other informal language such as "there is no way" or "it breaks the definition". Instead of "so" you could use "thus", "hence", "therefore", "it follows that", "which yields". Avoid the usage of "this" and "that" to refer to mathematical objects and instead write by name what is "this" and what is "that". For example, if the *it* in the phrase "it is empty" is the set S then write "**the set** S is empty" or "S is the empty set". Or instead of "it is a bjection" use "**the**

function f is a bijection". Thrive to use proper vocabulary, for example, instead of "these equations are **looking** to prove that ...", you could write "equations (1)-(3) will be used to prove that ...".

8. **When you are writing a proof sound confident** and avoid "we want to assume" or "let's assume". If you need to assume something then **assume it!** "**Assume** that f is a surjection ...", "**assume** that the claim holds for some $k \in \mathbb{N}$", and not "we want to assume that f is a surjection" or "now let's assume that the claim holds for some $k \in \mathbb{N}$".

9. **If you are asked to prove an identity, do not start your proof by writing down the identity and rearranging to show that two expressions are equal.** For example, if you are asked to prove that $(a+b)(a-b) = a^2 - b^2$ then do not do the following: "*We want to prove that* $(a+b)(a-b)$ *so,*

$$\begin{aligned}(a+b)(a-b) &= a^2 - b^2 \\ a^2 - ab + ba - b^2 &= a^2 - b^2 \\ a^2 - ab + ab - b^2 &= a^2 - b^2 \\ a^2 - b^2 &= a^2 - b^2\end{aligned}$$

and so $(a-b)(a+b) = a^2 - b^2$ *because it is true that* $a^2 - b^2 = a^2 - b^2$*.*" Instead, you could start with one side of the identity and perform the same algebraic steps: "*To prove the given identity we have that*

$$\begin{aligned}(a+b)(a-b) &= a^2 - ab + ba - b^2 \\ &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2.\end{aligned}$$

Notice that the last line of the equation has a period because we ended a sentence.

10. **Do not start an equation with an equal sign**. For example, consider the following:

"*Multiplying by* $a - bi$ *we get*

$$\begin{aligned}&= \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} \\ &= \frac{a-bi}{a^2+b^2} \\ &= \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2} i\end{aligned}$$

Instead, the writer meant the following:

"*Multiplying by* $a - bi$ *we get*

$$\begin{aligned} \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} &= \frac{a-bi}{a^2+b^2} \\ &= \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2} i. \end{aligned}$$

Again, notice the period used to end a sentence.

11. In a proof by induction, in the induction step you assume "that the claim holds for **some** $k \in \mathbb{N}$" and not "that the claim holds for **any** $k \in \mathbb{N}$" or not "that the claim holds for **all** $k \in \mathbb{N}$". If you assume that the claim holds for any or all $k \in \mathbb{N}$ then nothing remains to be proven.

12. Here are proper uses of "such that":

    (a) "there exists $p, q \in \mathbb{Q}$ such that $f(p, q) = 0$"
    (b) "the set S consists of all $x \in \mathbb{Q}$ such that $x^2 < 2$"
    (c) "there is an integer n such that $f(n) = 8$"
    (d) "if w is such that $2^w < y$ then"
    (e) "let z be a complex number such that $|z| = 1$"

    Almost always, "such that" is referring to an object that satisfies certain conditions. Improper uses of "such that" are:

    (a) "suppose that a set $S = \{x\}$ such that $\mathcal{P}(S) = \{\emptyset, \{x\}\}$"
    (b) "suppose we add an element b to S_k such that $S_{k+1} = S \cup \{b\}$"
    (c) "we will assume that $2^n < n!$ is true when $n = k$ such that $2^k < k!$"
    (d) "we also know that f is injective such that all elements in $\mathbb{Q}$..."

13. Review what it means for a function $f : A \to B$ to be injective. This is **not** the definition of an injection: "The function f is an injection if every element $x \in A$ has a unique image in B." This is the definition of a function.

14. Many of us, even trained mathematicians, **confuse or improperly use proof by contradiction with proof by the contrapositive**. As an example, consider the following statement: "Suppose that $f : Q \to P$ is a bijection. Prove that if Q is uncountable then P is uncountable." Here is a supposed proof by contradiction:

    "**Assume that Q is uncountable.** *Suppose by contradiction that P is countable. Then there exists an injection $g : P \to \mathbb{N}$. The composite mapping $(g \circ f) : Q \to \mathbb{N}$ is an* *injection and therefore Q is countable. This is a contradiction because we assumed that Q is uncountable. Therefore, P must be uncountable."*

Why is this not a proof by contradiction? Notice that the underlined assumption (that Q is uncountable) was never used in the proof and there is no contradiction. Instead, the proof is directly showing that "if P is countable then Q is countable" which is the contrapositive of "if Q is uncountable then P is uncountable". Whenever you are proving by contradiction you must use your assumption to arrive at some contradiction, perhaps something like $1 = 3$. A good example of this is the proof that "there exists no $x \in \mathbb{Q}$ such that $x^2 = 2$." One assumes by contradiction that there exists such $x = \frac{a}{b}$ with $\gcd(a, b) = 1$ and that $x^2 = 2$. One uses this assumption to arrive at the conclusion that $\gcd(a, b) \geq 2$ which is a contradiction.