Instructor: Cesar Aguilar
Department of Mathematics, SUNY Geneseo
South 325A, aguilar@geneseo.edu

Solutions to these questions will not be provided.

1. Differentiation

1. The function f is differentiable at c if _________________________________.
2. Use the definition of the derivative to find $f'(x)$ if $f(x) = \frac{1}{1+x}$. Clearly state the domain of $f'(x)$.
3. Prove by definition that the derivative of $f(x) = \sin(x)$ is $f'(x) = \cos(x)$ for all $x \in \mathbb{R}$.
4. Prove that $f(x) = |x|$ is not differentiable at $x = 0$.
5. Use the definition of the derivative to find $f'(x)$ and $f''(x)$ if
$$f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}.$$
In each case, clearly state the domain of the derivatives.
6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} x\cos(1/x^2), & x \neq 0 \\ 0, & x = 0. \end{cases}$$
Find the derivative function f', clearly stating its domain. Where is f' continuous?
7. Suppose that $f : [a, b] \to \mathbb{R}$ has a relative minimum at $c \in (a, b)$. Prove that if f is differentiable at c then $f'(c) = 0$.
8. State the Mean Value Theorem (MVT).
9. Recall that a function $f : A \to \mathbb{R}$ is Lipschitz on A if there exists $K > 0$ such **for all** $x, y \in A$ it holds that $|f(x) - f(y)| \leq K|x - y|$.
   (a) Use the Mean Value Theorem to prove that $f(x) = \arctan(x)$ is a Lipschitz function on $\mathbb{R}$.
   (b) Use the Mean Value Theorem to prove that $f(x) = \frac{1}{1+x^2}$ is Lipschitz on $\mathbb{R}$.
10. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).
    (a) Prove that if $f'(x)$ is **bounded** on (a, b) then f is a Lipschitz function on $[a, b]$.
    (b) Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is continuous on $[0, 1]$, differentiable on $(0, 1)$, but is not Lipschitz on $[0, 1]$.
11. Recall that $e \approx 2.71828\ldots$. Use Taylor's theorem at $x_0 = 0$ and the estimate $e < 3$ to show that for any $n \in \mathbb{N}$ it holds that
$$0 < e - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) < \frac{3}{(n+1)!}$$
(Notice that there are two inequalities that you have to prove here.)

2. Integration

1. Suppose that $f, g \in \mathcal{R}[a,b]$ and let $\alpha, \beta \in \mathbb{R}$. Prove by definition that $(\alpha f + \beta g) \in \mathcal{R}[a,b]$.

2. If f is Riemann integrable on $[a,b]$ and $|f(x)| \leq M$ for all $x \in [a,b]$, prove that $|\int_a^b f| \leq M(b-a)$. Hint: The inequality $|f(x)| \leq M$ is equivalent to $-M \leq f(x) \leq M$. Then use the fact that constants functions are Riemann integrable whose integrals are easily computed. Finally, apply a theorem from class.

3. If f is Riemann integrable on $[a,b]$ and $(\dot{\mathcal{P}}_n)$ is a sequence of tagged partitions of $[a,b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ prove that
$$\int_a^b f = \lim_{n\to\infty} S(f; \dot{\mathcal{P}}_n)$$
Hint: This is a neat question. For each $n \in \mathbb{N}$ we have the real number $x_n = S(f; \dot{\mathcal{P}}_n)$, and we therefore have a sequence (x_n). Let $L = \int_a^b f$. We therefore want to prove that $\lim\limits_{n\to\infty} x_n = L$. Use the definition of the limit of a sequence and apply the definition of Riemann integrability.

4. Give an example of a function $f : [0,1] \to \mathbb{R}$ that is Riemann integrable on $[c,1]$ for every $c \in (0,1)$ but which is not Riemann integrable on $[0,1]$. Hint: A Riemann integrable function must be bounded.

5. Although we did not get to prove this in class, it is true that **any continuous function** $f : [a,b] \to \mathbb{R}$ **is Riemann integrable on** $[a,b]$**.** Suppose then that $f : [a,b] \to \mathbb{R}$ is continuous and assume that $f(x) > 0$ for all $x \in [a,b]$. Prove that $\int_a^b f > 0$. Hint: A continuous function on a closed and bounded interval achieves its minimum value. Then apply the same theorem from class as you did in Problem 2. above.