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1. Comparison Tests

The direct Comparison Test has two parts.

Comparison Test. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences that are non-negative, in other words, $a_n \geq 0$ and $b_n \geq 0$. Suppose further that $a_n \leq b_n$ for all $n \geq 1$.

(i) If $\sum b_n$ converges then $\sum a_n$ also converges.

(ii) If $\sum a_n$ diverges then $\sum b_n$ also diverges.

The Comparison Test is very intuitive. For example, if $\sum b_n = B$ converges and $a_n \leq b_n$ then $\sum a_n \leq B$. So the sum $\sum a_n$ is finite and less than or equal to B. On the other hand, if $\sum a_n$ diverges then since $a_n \geq 0$ then $\sum a_n = \infty$. Therefore, since $a_n \leq b_n$ then the sum $\sum b_n$ also diverges. These are not precise mathematical statements but they capture the main idea. The proof of the Comparison Test would use partial sums of the sequences.

The truth about the Comparison Test: To be frank, the Comparison Test is a good method to apply when you already have a pretty good idea about whether the series you are given converges or diverges. If you have a pretty good idea that $\sum a_n$ converges, then to apply the Comparison test, you want to find a series $\sum b_n$ that you **know** converges and also that $a_n \leq b_n$. Similarly, if instead you have a pretty good idea that $\sum a_n$ diverges then you want to find a series $\sum b_n$ that you **know** diverges and also $a_n \leq b_n$. By now, you should have a list of series that you know converge or diverge.

Example 1: Determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n+2}{3n^2+1}$$

Solution: As a first test, we compute that

$$\lim_{n\to\infty} \left(\frac{n+2}{3n^2+1}\right) = 0,$$

and so we cannot conclude based on this that the series diverges. Now, when n is very large, $n+2 \approx n$ and $3n^2+1 \approx 3n^2$. Thus, when n is very large

$$\frac{n+2}{3n^2+1} \approx \frac{n}{3n^2} = \frac{1}{3n}$$

We know that $\sum_{n=1}^{\infty} \frac{1}{3n}$ diverges because it is a constant multiple of the Harmonic series. So, we have a good idea that the series we are given probably diverges. Let's now be precise and find a sequence

c_n such that $c_n \le \frac{n+2}{3n^2+1}$ and such that $\sum c_n$ diverges. First of all, $n < n+2$ and therefore

$$\frac{n}{3n^2+1} < \frac{n+2}{3n^2+1} \qquad \text{Inequality (1)}$$

Now, $3n^2+1 \le 3n^2+n^2$ for all $n = 1, 2, 3, \ldots$. Then, $\frac{1}{3n^2+n^2} \le \frac{1}{3n^2+1}$ for all n, and multiplying both sides by n we get

$$\frac{n}{3n^2+n^2} \le \frac{n}{3n^2+1} \qquad \text{Inequality (2)}$$

Combining Inequality (1) and (2), we get

$$\frac{n}{3n^2+n^2} \le \frac{n}{3n^2+1} < \frac{n+2}{3n^2+1}.$$

Now, $\dfrac{n}{3n^2+n^2} = \dfrac{n}{4n^2} = \dfrac{1}{4n}$ and thus

$$\frac{1}{4n} \le \frac{n}{3n^2+1}.$$

The series $\sum \frac{1}{4n}$ diverges and therefore by Part (ii) of the Comparison test, $\sum \frac{n}{3n^2+1}$ also diverges.

Example 2: Determine if the given sequence is convergent of divergent.

$$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$$

Solution: We first compute that $\lim_{n\to\infty} \frac{1}{n+\sqrt{n}} = 0$, and so the sequence might be convergent. Now, when n is very (very) large, $\sqrt{n}$ is very small compared to n. So, when n is large

$$\frac{1}{n+\sqrt{n}} \approx \frac{1}{n}.$$

So, we have reason to believe that $\sum \frac{1}{n+\sqrt{n}}$ is divergent. To be precise about this, it is clear that $\sqrt{n} \le n$ for all $n = 1, 2, 3, \ldots$. Therefore, $n + \sqrt{n} \le n + n = 2n$, and therefore,

$$\frac{1}{2n} \le \frac{1}{n+\sqrt{n}}.$$

The series $\sum \frac{1}{2n}$ is divergent, and therefore by Part (ii) of the Comparison Test, the series $\sum \frac{1}{n+\sqrt{n}}$ is also divergent.

Example 3: Determine if the given sequence converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n3^n}.$$

Solution: The denominator $n3^n$ grows faster than 3^n and we know that the Geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges. So, we have good reason to believe that the given series converges. To see this, we first note that $3^n \le n3^n$ for $n = 1, 2, 3, \ldots$. Thus,

$$\frac{1}{n3^n} \le \frac{1}{3^n}$$

The series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent Geometric series, and thus by Part (i) of the Comparison Test, the given series $\sum_{n=1}^{\infty} \frac{1}{n3^n}$ is also convergent.

From the examples above, we can see that applying the Comparison Test takes some ingenuity and insight. The next test for convergence/divergence, called the Limit Comparison Test, is a direct application of the Comparison Test but is a bit more easy to apply provided you can do limits. The Limit Comparison Test is particularly useful for series $\sum a_n$ where a_n is a rational function of n.

The Limit Comparison Test. Suppose that $a_n > 0$ and $b_n > 0$ for all n. Compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

(i) If $c > 0$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

(ii) If $c = 0$ and $\sum b_n$ converges then $\sum a_n$ converges.

(iii) If $c = \infty$ and $\sum b_n$ diverges then $\sum a_n$ diverges.

Let's now use the Limit Comparison Test to redo the three previous examples where we used the Comparison Test. You can decide which test is easier to apply.

Example 4: Determine if the given series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n+2}{3n^2+1}$$

Solution: Here $a_n = \frac{n+2}{3n^2+1}$. When n is very large, $\frac{n+2}{3n^2+1} \approx \frac{n}{3n^2} = \frac{1}{3n}$, and we know that $\sum \frac{1}{3n}$ diverges. So we have good reason to believe the given series diverges. Let $b_n = \frac{1}{3n}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{n+2}{3n^2+1}}{\frac{1}{3n}} \\
&= \lim_{n\to\infty} \frac{(n+2)3n}{3n^2+1} \\
&= \lim_{n\to\infty} \frac{3n^2+6n}{3n^2+1} \\
&= 1
\end{aligned}$$

Therefore, since $\sum \frac{1}{3n}$ diverges, then by Part (i) of the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{n+2}{3n^2+1}$ diverges also.

Example 5: Determine if the given sequence is convergent of divergent.

$$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$$

Solution: Here $a_n = \frac{1}{n+\sqrt{n}}$. When n is very large, $\sqrt{n}$ is much smaller than n, so $n + \sqrt{n} \approx n$. Hence, when n is large, $a_n = \frac{1}{n+\sqrt{n}} \approx \frac{1}{n}$, and we know that $\sum \frac{1}{n}$ diverges. Now,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{n}{n+\sqrt{n}} \\
&= \lim_{n\to\infty} \frac{n}{n(1+\frac{1}{\sqrt{n}})} \\
&= \lim_{n\to\infty} \frac{1}{1(1+\frac{1}{\sqrt{n}})} \\
&= 1 \qquad \text{could've used L.H.R. also}
\end{aligned}$$

Therefore, since $\sum \frac{1}{n}$ diverges, then by Part (i) of the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ also diverges.

Example 6: Determine if the given sequence converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n3^n}.$$

Solution: Here $a_n = \frac{1}{n3^n}$. Let $b_n = \frac{1}{3^n}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{n3^n}}{\frac{1}{3^n}} \\
&= \lim_{n\to\infty} \frac{3^n}{n3^n} \\
&= \lim_{n\to\infty} \frac{1}{n} \\
&= 0
\end{aligned}$$

Therefore, since $\sum \frac{1}{3^n}$ converges, then by Part (ii) of the Limit Comparison Test, $\sum \frac{1}{n3^n}$ also converges.

Example 7: Determine if the given sequence converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n + n}{7n2^n}$$

Solution: Here $a_n = \frac{2^n+n}{7n2^n}$. Let's first test for divergence:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{2^n + n}{7n2^n} \\
&= \lim_{n\to\infty} \frac{2^n(1+\frac{n}{2^n})}{7n2^n} \\
&= \lim_{n\to\infty} \frac{(1+\frac{n}{2^n})}{7n} \\
&= 0
\end{aligned}$$

Thus, we cannot conclude that the given series diverges and we need to do further analysis. Now, when n is large $2^n + n \approx 2^n$ because exponentials grow much faster than polynomials. So, when n is large $a_n = \frac{2^n+n}{7n2^n} \approx \frac{2^n}{7n2^n} = \frac{1}{7n}$. We know that $\sum \frac{1}{7n}$ diverges so we have good reason to believe that the given series diverges. Let $b_n = \frac{1}{7n}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{2^n+n}{7n2^n}}{\frac{1}{7n}} \\
&= \lim_{n\to\infty} \frac{(2^n+n)7n}{7n2^n} \\
&= \lim_{n\to\infty} \frac{2^n+n}{2^n} \\
&= \lim_{n\to\infty} \frac{2^n(1+\frac{n}{2^n})}{2^n} \\
&= \lim_{n\to\infty} \left(1+\frac{n}{2^n}\right) \\
&= 1
\end{aligned}$$

Therefore, since $\sum \frac{1}{7n}$ diverges, then by Part (i) of the Limit Comparison Test, the given series $\sum_{n=1}^{\infty} \frac{2^n+n}{7n2^n}$ also diverges.

Example 8: Determine if the given sequence converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$$

Solution: Here $a_n = \frac{(\ln n)^2}{n^3}$. Let's first test for divergence:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{(\ln n)^2}{n^3} && \frac{\infty}{\infty} \text{ so use L.H.R.} \\
&= \lim_{n\to\infty} \frac{2(\ln n)\frac{1}{n}}{3n^2} \\
&= \lim_{n\to\infty} \frac{2(\ln n)}{3n^3} && \frac{\infty}{\infty} \text{ again so use L.H.R.} \\
&= \lim_{n\to\infty} \frac{2\frac{1}{n}}{9n^2} \\
&= \lim_{n\to\infty} \frac{2}{9n^3} \\
&= 0
\end{aligned}$$

So, we cannot conclude that the series diverges and we need to do further analysis. **A useful fact that we can use is that for every $n > 0$, it holds that $\ln(n) < \sqrt{n}$.** The graph of $f(x) = \ln(x)$ and $g(x) = \sqrt{x}$ on the interval $[1, 10]$ are shown in Figure 1.

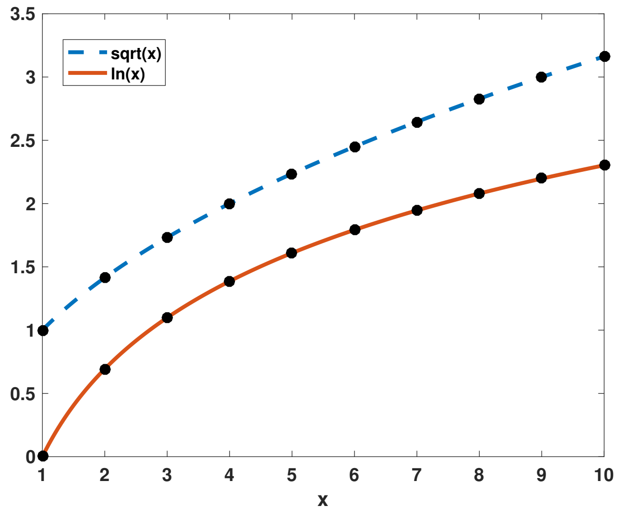

Figure 1: Graph of $f(x) = \ln(x)$ (red-solid graph) and $g(x) = \sqrt{x}$ (blue-dashed graph)

Therefore, since $\ln(n) < \sqrt{n}$ then squaring both sides yields $(\ln n)^2 < n$. Therefore,

$$a_n = \frac{(\ln n)^2}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}.$$

The series $\sum \frac{1}{n^2}$ converges, and therefore, by the Comparison Test Part (i), the given series $\sum \frac{(\ln n)^2}{n^3}$ also converges.

Example 9: Determine if the given series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

Solution: The given series is a p-series $\sum \frac{1}{n^p}$ with $p = \frac{1}{2}$. We know that a p-series converges only if $p > 1$, so this series diverges. Let's use the Limit Comparison test to show this. The series $\sum \frac{1}{n}$ diverges and

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{1}{n^{1/2}}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{n}{n^{1/2}} \\ &= \lim_{n\to\infty} n^{1/2} \\ &= \infty \end{aligned}$$

Therefore, by Part (iii) of the Limit Comparison test, the series $\sum \frac{1}{n^{1/2}}$ diverges. In fact, this same procedure shows that if $0 < p < 1$ then the p-series $\sum \frac{1}{n^p}$ diverges because

$$\lim_{n\to\infty} \frac{\frac{1}{n^p}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^p} = \lim_{n\to\infty} n^{1-p} = \infty$$

whenever $1 - p > 0$, which is the same as saying that $p < 1$.

2. Absolute Convergence Test

Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, so that here $a_n = \frac{(-1)^n}{n^2}$. This is an example of an **alternating series**; the terms of the series alternate between positive and negative. This series is not Geometric, we cannot apply the Integral Test because some of the series terms are negative (go read the Integral Test to see), and we cannot apply the comparison tests because they require that the series terms also be non-negative (go read the comparison tests to see). However, we know that the series of the absolute values $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and so we might use this fact to conclude that the original series $\sum_{n=1}^{\infty} a_n$ converges. This is in fact true and it is called the Absolute Convergence Test.

Absolute Convergence Test. Let $\sum a_n$ be a given series. If the series of absolute values $\sum |a_n|$ converges then $\sum a_n$ also converges.

The Absolute Convergence Test is useful when some of the terms of the series $\sum a_n$ are negative, which is the case for alternating series. Because the series $\sum |a_n|$ has non-negative terms, we may use the Integral Test or the Comparison Tests on the series $\sum |a_n|$ to show that it converges, if possible.

Example 10: Determine whether the given series converges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Solution: Here $a_n = \frac{(-1)^n}{n^2}$ and thus $|a_n| = \frac{|(-1)^n|}{|n^2|} = \frac{1}{n^2}$. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Therefore, by the Absolute Convergence test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ also converges.

A series $\sum a_n$ for which $\sum |a_n|$ converges is somewhat special; not only does $\sum |a_n|$ converge but by the Absolute Convergence Test, $\sum a_n$ also converges. Hence, for such a series, we will say that $\sum a_n$ is **absolutely convergent** or that it **converges absolutely**.

Example 11: Determine whether the given series converges.

$$\sum_{n=0}^{\infty} \frac{8(-1)^n}{(3+(1/n))^{2n}}$$

Solution: The series contains negative terms so we cannot apply the Integral Test or any of the Comparison Tests. It is clear that $\lim_{n\to\infty} a_n = 0$, so we cannot conclude that the series diverges. Consider the absolute value series

$$\sum_{n=0}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{8}{(3+(1/n))^{2n}}$$

When n is large $\frac{1}{n} \approx 0$, and so the terms of the series behave like

$$\frac{8}{(3+(1/n))^{2n}} \approx \frac{8}{(3+0)^{2n}} = \frac{8}{3^{2n}}$$

We know that the series $\sum \frac{8}{3^{2n}}$ converges (it is Geometric with $r = \frac{1}{3^2}$) and so we have good reason to believe that $\sum |a_n|$ converges. In fact, by direct comparison

$$\frac{8}{(3 + (1/n))^{2n}} < \frac{8}{3^{2n}}$$

The series $\sum \frac{8}{3^{2n}}$ converges and therefore by the Comparison Test Part (i), the series $\sum |a_n| = \sum \frac{8}{(3+(1/n))^{2n}}$ also converges. Thus, by the Absolute Convergence test, the series $\sum a_n = \sum \frac{8(-1)^n}{(3+(1/n))^{2n}}$ converges also.

Example 12: Determine whether the given series converges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$$

Solution: This is an alternating Harmonic series. The absolute value series $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges but we cannot conclude that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ diverges **because the Absolute Convergence Test is a test for convergence and not for divergence**. Thus, we cannot conclude anything about the given series with the known tests thus far.

Example 13: Determine whether the given series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$$

Solution: Because $\cos(n)$ takes on both negative and positive values for $n = 1, 2, 3, \ldots$, we cannot apply the Integral Test or the Comparison Test. Using the Squeeze Theorem, we can show that

$$\lim_{n \to \infty} \frac{\cos(n)}{n^2} = 0$$

and therefore we cannot conclude that the series diverges. Consider the series of absolute values $\sum |a_n| = \sum \frac{|\cos(n)|}{n^2}$. For all $n = 1, 2, 3, \ldots$, it holds that $|\cos(n)| \leq 1$ and therefore dividing this inequality by n^2 we obtain

$$\frac{|\cos(n)|}{n^2} \leq \frac{1}{n^2}$$

The series $\sum \frac{1}{n^2}$ converges and therefore by Part (i) of the Comparison Test, the series $\sum \frac{|\cos(n)|}{n^2}$ converges. Therefore, the series $\sum \frac{\cos(n)}{n^2}$ is absolutely convergent.

3. The Ratio Test

The Ratio Test can be applied to any series, it does not matter if it takes on negative values. However, the test is not full proof; in some cases it is inconclusive. The Ratio Test is also very useful when factorials $n!$ are present in the series.

The Ratio Test. Let $\sum a_n$ be any given series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$$

The following hold:

(a) If $\rho < 1$ then the series $\sum a_n$ converges absolutely.

(b) If $\rho > 1$ or if $\rho = \infty$ then the series $\sum a_n$ diverges.

(c) If $\rho = 1$ then the test is inconclusive; it may converge or it may diverge and further analysis is necessary.

Example 14: Determine whether the given series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{2^n}{n!}$$

Solution: Here $a_n = \frac{2^n}{n!}$ and therefore $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right| && \text{all terms are positive}\\
&= \lim_{n\to\infty}\frac{2^{n+1}n!}{2^n(n+1)!}\\
&= \lim_{n\to\infty}\frac{2^n\cdot 2\cdot n!}{2^n(n+1)\cdot n!}\\
&= \lim_{n\to\infty}\frac{2}{n+1}\\
&= 0
\end{aligned}$$

Therefore, since $\rho = 0 < 1$, by the Ratio Test, the given series is convergent.

Example 15: Determine whether the given series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{n^3}{e^n}$$

Solution: Here $a_n = \frac{n^3}{e^n}$ and therefore $a_{n+1} = \frac{(n+1)^3}{e^{n+1}}$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^3}{e^{n+1}} \div \frac{n^3}{e^n}\right| \\
&= \lim_{n\to\infty}\frac{(n+1)^3e^n}{e^{n+1}n^3} \\
&= \lim_{n\to\infty}\frac{(n+1)^3}{e\cdot n^3} \\
&= \lim_{n\to\infty}\frac{n^3+3n^2+3n+1}{e\cdot n^3} \\
&= \frac{1}{e}.
\end{aligned}$$

Since $\rho = \frac{1}{e} < 1$, by the Ratio Test, the given series converges.

Example 16: Determine whether the given series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{n^2 2^n(-1)^n}{3^n}$$

Solution: Here $a_n = \frac{n^2 2^n(-1)^n}{3^n}$ and therefore $a_{n+1} = \frac{(n+1)^2 2^{n+1}(-1)^{n+1}}{3^{n+1}}$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^2 2^{n+1}(-1)^{n+1}}{3^{n+1}} \div \frac{n^2 2^n(-1)^n}{3^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)^2 2^{n+1}(-1)^{n+1}}{3^{n+1}} \times \frac{3^n}{n^2 2^n(-1)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)^2\cdot 2\cdot(-1)}{3n^2}\right| \\
&= \lim_{n\to\infty}\frac{2(n+1)^2}{3n^2} \\
&= \frac{2}{3}
\end{aligned}$$

Therefore, since $\rho = \frac{2}{3} < 1$, by the Ratio Test the given series converges.

Example 17: Determine whether the given series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{(-1)^n(2n)!}{n!(n+1)!}$$

Solution: Here $a_n = \frac{(-1)^n(2n)!}{n!(n+1)!}$ and therefore

$$a_{n+1} = \frac{(-1)^{n+1}(2(n+1))!}{(n+1)!((n+1)+1)!} = \frac{(-1)^{n+1}(2n+2)!}{(n+1)!(n+2)!}.$$

Therefore,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(2n+2)!}{(n+1)!(n+2)!} \div \frac{(-1)^n(2n)!}{n!(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(2n+2)!}{(n+1)!(n+2)!} \times \frac{n!(n+1)!}{(-1)^n(2n)!}\right|$$

$$= \lim_{n\to\infty}\frac{(2n+2)!\cdot n!}{(n+2)!\cdot(2n)!}$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)(2n)!\cdot n!}{(n+2)(n+1)n!\cdot(2n)!}$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+2)(n+1)}$$

$$= \lim_{n\to\infty}\frac{4n^2+4n+2}{n^2+3n+2}$$

$$= 4$$

Therefore, by the Ratio Test, since $\rho = 4 > 1$, the given series diverges. Note that here we used that fact that $(2n+2)! = (2n+2)(2n+1)(2n)!$ and similarly that $(n+2)! = (n+2)(n+1)n!$.

Example 18: Determine whether the given series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{n!}{n^n}$$

Solution: It is not clear whether $\lim_{n\to\infty}\frac{n!}{n^n} = 0$ because $n!$ grows very fast but n^n grows faster. In fact, for $n \geq 2$, it holds that $n! < n^n$ and therefore $\frac{n!}{n^n} < 1$ for $n = 2, 3, \ldots,$, and so it is possible that $\lim_{n\to\infty}\frac{n!}{n^n} = 0$. To see that $n! < n^n$:

$$n! = 1\cdot 2\cdot 3\cdots n < \underbrace{n\cdot n\cdot n\cdots n}_{n\text{ times}} = n^n$$

Anyways, let's try the Ratio Test. Here $a_n = \frac{n!}{n^n}$ and therefore $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$. Therefore,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{n+1}} \div \frac{n!}{n^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)n!n^n}{(n+1)^n(n+1)n!}$$

$$= \lim_{n\to\infty}\frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n \qquad \text{get indeterminate power } 1^\infty$$

The last limit gives the indeterminate power 1^∞, so we consider

$$\begin{aligned}
\lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right)^n &= \lim_{n\to\infty} n\ln\left(\frac{n}{n+1}\right) && \text{get } \infty\cdot 0\text{, so rewrite}\\
&= \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} && \text{now get } \frac{0}{0}\text{, so apply L.H.R.}\\
&= \lim_{n\to\infty} \frac{\frac{1}{n}-\frac{1}{n+1}}{-\frac{1}{n^2}}\\
&= \lim_{n\to\infty} \frac{-n^2}{n(n+1)}\\
&= -1
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = e^{-1} = \frac{1}{e}.$$

Hence, since $\rho = \frac{1}{e} < 1$, then by the Ratio Test, the given series converges. Note that this means that indeed we have that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$.